Coexistence in Wireless Decentralized Networks in the Low SNR Regime: To Hop or Not to Hop?

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Abstract

We consider a wireless communication network with a fixed number u of frequency sub-bands to be shared among N transmitter-receiver pairs. In traditional frequency division (FD) systems, the available band is partitioned into disjoint clusters (frequency sub-bands) and assigned to different users (each user transmits only in its own cluster). If the number of users sharing the spectrum is random, this technique may lead to inefficient spectrum utilization (a considerable fraction of the sub-bands may remain empty most of the time). In addition, this approach inherently requires either a central network controller for frequency allocation, or cognitive radios which sense and occupy the empty sub-bands in a dynamic fashion. These shortcomings motivate us to look for a decentralized scheme (without using cognitive radios) which allows the users to coexist, while utilizing the spectrum efficiently. A frequency hopping (FH) scheme (with $\mathrm{i.i.d.}$ Gaussian code-books) is already proposed in [] where each user transmits over a selection of sub-bands and hops to another selection (with the same cardinality) from transmission to transmission. It is shown that in higher ranges of SNR, frequency hopping offers considerable improvement in terms of various measures such as average sum-rate multiplexing gain and the so called " ϵ-outage capacity". In this article, we rise the question if hopping is optimum for all ranges of SNR. We consider two different scenarios. In the first scenario, we consider a wireless network where the absolute value of all the forward channel gains is more than a threshold ϵ_1 and the absolute value of all the crossover gains is less than a threshold ϵ_2. We show that as far as $\frac{\epsilon_2}{\epsilon_1} < \frac{1}{\sqrt{N-1}}$, there is a γ_0 such that if $\mathrm{SNR} \leq \gamma_0$, the sum-rate of the system is maximized if all users spread their power on the whole spectrum. In particular, if $N = 2$, we prove $\gamma_0 \geq \frac{\sqrt{5}-1}{4}u$. In the sequel, we consider the case where the fading coefficients and the number of active users in the system are unknown to

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transmitters. Via computing the so called ϵ-outage capacity, we demonstrate that for sufficiently low SNR values, hopping has no advantage over the case where all users spread their power on the whole spectrum.

I. Introduction

Optimal resource allocation is an imperative issue in wireless networks. When multiple users share the same spectrum, the destructive effect of multi-user interference can limit the achievable rates. As such, an effective and low complexity frequency sharing strategy which maximizes the degrees of freedom per user while mitigating the impact of the multi-user interference is desirable. In frequency division (FD) systems, different users transmit over disjoint frequency sub-bands. Due to practical considerations, such FD systems usually rely on a fixed number of such frequency sub-bands. The main drawback of FD systems is that most of the time the majority of the potential users may be inactive, reducing the resulting spectral efficiency. Reference [1] considers a network of several users with mutual interference. Treating the interference as noise, a central controller computes the optimum power allocation of each link over the spectrum to maximize a global utility function. This leads to the best spectrum sharing strategy for a specific number of users. Clearly, if the number of users changes, the system is not guaranteed to offer the best possible spectral efficiency. In fact, it is shown in [1] that if the crossover gains are sufficiently greater than the forward gains, the frequency division is optimum. However, as mentioned earlier, if the number of users sharing the spectrum is random, FD systems can be highly inefficient in terms of the overall spectral efficiency. To avoid the need for a central controller, cognitive radios [2] are introduced which can sense the bands and transmit over an unoccupied portion of the available spectrum. Fundamental limits of wireless networks with cognitive radios are studied in [3]–[7]. Although cognitive radios avoid the use of a central controller, they require methods for frequency sensing and dynamic frequency assignment which add to the overall system complexity. For example, in opportunistic communication, each cognitive device must search for idle regions of the spectrum or spectrum holes which requires sophisticated detection techniques [8]–[10]. On the other hand, in both game-theoretic scenarios and cognitive radios, randomness of the number of users is not taken into account. Noting the above points, it is desirable to have a decentralized frequency sharing strategy (without the need for cognitive radios) which allows the users to coexist, while utilizing the spectrum efficiently and fairly.

Being a standard technique in spread spectrum communications and due to its interference avoidance nature, frequency hopping is the simplest spectrum sharing method to use in decentralized networks. As different users typically have no prior information about the codebooks of the other users, the most efficient method (specially in higher ranges of SNR) is avoiding interference by choosing unused channels. As mentioned earlier, searching the spectrum to find spectrum holes is not an easy task due to the dynamic spectrum usage. As such, frequency hopping is a realization of transmission without sensing while avoiding the collisions as much as possible.

Frequency hopping is one of the standard signaling schemes [15] adopted in ad-hoc networks. In short range scenarios, bluetooth systems [19]–[21] are the most popular examples of a wireless personal area network or WPAN. By using frequency hopping over the unlicensed ISM band, a bluetooth system provides robust communication against unpredictable sources of interference. A modification of frequency hopping called dynamic frequency hopping (DFH), selects the hopping pattern based on interference measurements in order to avoid dominant interferers. The performance of the DFH scheme when applied to a cellular system is assessed in [22]–[24]. Frequency hopping is also proposed in [7] in the context of cognitive radios where each cognitive transmitter selects a frequency band but quits transmitting if the band is already occupied by a primary user.

Already in [55], motivated by the fact that frequency hopping leaves a portion of the spectrum clean, we have considered a decentralized party of N users sharing u discrete frequency sub-bands via frequency hopping. Different transmitters are linked to different receivers through paths with static and non-frequency-selective fading. Each user is assumed to have no prior knowledge about the code-books of the other users. We proposed a frequency hopping (FH) strategy in which the i^{th} user selects v_i frequency sub-bands among the u available sub-bands and hops to another set of v_i sub-bands for the next transmission. It is assumed that all users transmit independent Gaussian code-books over their chosen frequency sub-bands.

As each user hops over different subsets of the sub-bands without informing other users about its hopping pattern, sensing the spectrum to track the instantaneous interference is a difficult task. This assumption makes the interference probability density function (PDF) on each frequency sub-band at the receiver side of each user be mixed Gaussian. Since the channel gains have a continuous PDF, the number of Gaussian components in the interference PDF on each frequency sub-band is 2^{N-1} with probability one. It is presumed that each user is able to derive the interference PDF after a sufficiently long training period at the receiver side.

It is already shown [54], [55] that FH outperforms FD in terms of different performance measures such as average sum-rate multiplexing gain (in case all the channel gains and the number of users are revealed to transmitters) and the so called ϵoutage capacity (in case the channel gains and the number of active users are unknown to transmitters). However, these results are valid for higher ranges of SNR. This paper deals with the case where SNR is low, i.e., the results are valid under a certain level of SNR.

We consider two different categories. In the first scenario, we consider a wireless network where the absolute value of all the forward channel gains is more than a threshold ϵ_1 and the absolute value of all the crossover gains is less than a threshold ϵ_2. We show that as far as $\frac{\epsilon_2}{\epsilon_1} < \frac{1}{\sqrt{N-1}}$, there is a γ_0 such that if $\mathrm{SNR} \leq \gamma_0$, the sum-rate of the system is maximized if all users spread their power on the whole spectrum. In particular, if $N = 2$, we prove $\gamma_0 \geq \frac{\sqrt{5}-1}{4}u$. In the sequel, we consider the case where the fading coefficients and the number of active users in the system are unknown to transmitters. Via computing the so called ϵ-outage capacity, we demonstrate that for sufficiently low SNR values, hopping has no advantage over the case where all users spread their power on the whole spectrum.

The paper outline is as follows. System model is given in section II. Sections III and IV are devoted to derive lower and upper bounds on the achievable rates of users respectively. Finally, section V deals with characterizing the hopping strategy in the low SNR regime.

II. System Model

We consider a communication system with N users[1] where the i^{th} user exploits $v \leq u$ out of the u sub-bands and spreads its available power, P, equally over these selected sub-bands by transmitting Gaussian signals of variance $\frac{P}{v}$ and mutual correlation coefficient ρ_i over the v chosen bands. The i^{th} user selects ρ_i according to a probability density function $f(\rho)$ over $[0,1]$. The function $f(\rho)$ is taken to be globally known to all users. This user hops to another set of v frequency sub-bands after each transmission. We denote the achievable rate of the i^{th} user by $\mathscr{R}_i$. The static and non frequency-selective fading coefficient of the link connecting the i^{th} transmitter to the j^{th} receiver is shown by $h_{i,j}$. Each receiver knows already the hopping pattern of its affiliated transmitter. On the other hand, as all users hop over different portions of the spectrum from transmission to transmission, no user is assumed to be capable of tracking the instantaneous interference. This assumption makes the interference plus noise PDF at the receiver side of each user be a mixed Gaussian distribution. In fact, depending on different choices the other users make to select the frequency sub-bands and values of the crossover gains, the interference

[1] Each user consists of a transmitter-receiver pair.

on each frequency sub-band at the receiver side of any user has up to 2^{N-1} power levels. For each i, the channel model for the i^{th} user is as follows:

$$\vec{Y}_i = h_{i,i}\vec{X}_i + \vec{Z}_i \tag{1}$$

where $\vec{X}_i$ is the $u \times 1$ input vector of the i^{th} user and $\vec{Z}_i$ is the noise plus interference vector on the receiver side of the i^{th} user. One may write $p_{\vec{X}_i}(\vec{x}) = \sum_{C \in \mathcal{C}} \frac{1}{\binom{u}{v}} g(\vec{x}, C)$ where $g(\vec{x}, C)$ denotes a zero-mean jointly Gaussian distribution of covariance matrix C and the set $\mathcal{C}$ includes all $u \times u$ diagonal matrices where v out of the u diagonal elements are $\frac{P}{v}$ while the rest are zeros. Denoting the noise plus interference on the j^{th} sub-band at the receiver side of the i^{th} user by $Z_{i,j}$ (the j^{th} component of $\vec{Z}_i$), it is clear that $p_{Z_{i,j}}(z)$ is not dependent on j. This is by the fact that crossover gains are not sensitive to frequency and there is no particular interest in a specific frequency sub-band by any user. We assume there are $L_i + 1$ ($L_i \leq 2^{N-1} - 1$) possible non-zero power levels for $Z_{i,j}$, say $\{\sigma_{i,l}^2\}_{l=0}^{L_i}$. The occurrence probability of $\sigma_{i,l}^2$ is denoted by $a_{i,l}$. Then, $p_{Z_{i,j}}(z)$ is a mixed Gaussian distribution as follows:

$$p_{Z_{i,j}}(z) = \sum_{l=0}^{L_i} \frac{a_{i,l}}{\sqrt{2\pi}\sigma_{i,l}} \exp -\frac{z^2}{2\sigma_{i,l}^2} \tag{2}$$

where $\sigma^2 = \sigma_{i,0}^2 < \sigma_{i,1}^2 < \sigma_{i,2}^2 < \ldots < \sigma_{i,L_i}^2$ (σ^2 is the ambient noise power). In fact, one may write $Z_{i,j} = \sum_{\substack{k=1 \\ k \neq i}}^{N} \epsilon_{k,j} h_{k,i} X_{k,j} + \nu_{i,j}$ where $X_{k,j}$ is the signal of the k^{th} user sent on the j^{th} sub-band, $\epsilon_{k,j}$ is a Bernoulli random variable showing if the k^{th} user has utilized the j^{th} sub-band and $\nu_{i,j}$ is the ambient noise which is a zero-mean Gaussian random variable with variance σ^2. Obviously, $\Pr\{\epsilon_{k,j} = 1\} = \frac{v}{u}$. To compute $\mathscr{R}_i$, one may see that for each i, the communication channel of the i^{th} user is a channel with state S_i, the hopping pattern, which is independently changing over different transmissions and is known to both the transmitter and receiver ends of the i^{th} user. The achievable rate of such a channel is given by

$$\mathscr{R}_i = I(\vec{X}_i; \vec{Y}_i | S_i) = \sum_{s_i \in \mathcal{S}_i} \Pr(S_i = s_i) I(\vec{X}_i; \vec{Y}_i | S_i = s_i) \tag{3}$$

where $I(\vec{X}_i; \vec{Y}_i | S_i = s_i)$ is the mutual information between $\vec{X}_i$ and $\vec{Y}_i$ for the specific sub-band selection dictated by $S_i = s_i$. The set $\mathcal{S}_i$ denotes all possible selections of v_i out of the u sub-bands. As $p_{\vec{Z}_i}(\vec{z})$ is a symmetric density function, meaning all its components have the same PDF given in (2), we deduce that $I(\vec{X}_i; \vec{Y}_i | S_i = s_i)$ is independent of s_i. Therefore, we may assume any specific sub-band selection for the

i^{th} user in $\mathcal{S}_i$, say the first v_i out of the u sub-bands. Denoting this specific state by s_i^*, we get:

$$\mathscr{R}_i = I(\vec{X}_i; \vec{Y}_i | S_i = s_i^*). \tag{4}$$

In this case, we denote $\vec{Y}_i$ and $\vec{X}_i$ by $\vec{Y}_i(s_i^*)$ and $\vec{X}_i(s_i^*)$ respectively. Obviously, we have:

$$\mathscr{R}_i = I(\vec{X}_i(s_i^*); \vec{Y}_i(s_i^*)) = \mathrm{h}(\vec{Y}_i(s_i^*)) - \mathrm{h}(\vec{Z}_i). \tag{5}$$

According to the system model proposed before, one may write:

$$\vec{Z}_i = \sum_{\substack{k=1 \\ k \neq i}}^{N} \vec{\xi}_{k,i} + \vec{\eta}_i \tag{6}$$

where $\vec{\xi}_{k,i}$ is the mixed gaussian interference vector imposed by the k^{th} user at the receiver side of the i^{th} user. Based on the specifications of the interference model given in the previous section, we write $\vec{\xi}_{k,i}$ as follows:

$$\vec{\xi}_{k,i} = h_{k,i} \vec{\xi}'_k \tag{7}$$

where $\vec{\xi}'_k$ is a random vector of mixed Gaussian distribution where each gaussian component of it corresponds to a specific occupation of v frequency bands. For example, for $u = 2$ and $v = 1$, it has the following distribution:

$$p_{\vec{\xi}'_k}(a, b) = \frac{1}{2\sqrt{2\pi P}} \left(\delta(b) \exp - \frac{a^2}{2P} + \delta(a) \exp - \frac{b^2}{2P} \right) \tag{8}$$

where $\delta(.)$ is the Dirac delta function. Clearly, $\vec{\xi}'_k$ is i.i.d. over k. The achievable rate of the i^{th} user, $\mathscr{R}_i$, is given in (5). In the following sections, we derive appropriate upper and lower bounds on $\mathscr{R}_i$ which enable us to partially characterize the low SNR regime optimal spectrum sharing rules. The bounds derived here are different from those obtained in the previous chapter, as the bounds in chapter 2 are useful in the high SNR regime and are loose in the low SNR case. On the other hand, the bounds obtained in this chapter are well suited to study the low SNR case and are loose in the high SNR regime.

III. Lower Bounds on The Achievable Rates

If we simply replace $\vec{Z}_i$ by a gaussian vector of the same covariance matrix, the mutual information decreases [20]. As such, we have:

$$\mathscr{R}_i \geq \frac{1}{2}\log\frac{\det(C(\vec{Y}_i(s_i^*)))}{\det(C(\vec{Z}_i))}. \tag{9}$$

On the other hand, denoting $C(\vec{\xi}_{k,i})$ by $C_{k,i}$, we have:

$$C(\vec{Z}_i) = \sum_{\substack{k=1\\k\neq i}}^{N} C(\vec{\xi}_{k,i}) + C(\vec{\eta}_i) = \sum_{\substack{k=1\\k\neq i}}^{N} C_{k,i} + \sigma^2 I_u. \tag{10}$$

We have:

$$C_{k,i} = |h_{k,i}|^2 \mathrm{E}\{\vec{\xi'_k}\vec{\xi'}^T_k\}. \tag{11}$$

To compute $\mathrm{E}\{\vec{\xi'_k}\vec{\xi'}^T_k\}$, we proceed as follows. Let us denote the j^{th} element of $\vec{\xi'_k}$ by $\xi'_k(j) = X_{k,j}\epsilon_{k,j}$. We have:

$$\mathrm{E}\{\xi'_k(j)^2\} = \int_0^1 \mathrm{E}\{\xi'_k(j)^2|\rho_k=\rho\}f(\rho)d\rho = \frac{P}{v}\Pr\{\epsilon_{k,j}=1\} \tag{12}$$

and

$$\begin{aligned}\mathrm{E}\{\xi'_k(j)\xi'_k(j')\}|_{j\neq j'} &= \int_0^1 \mathrm{E}\{\xi'_k(j)\xi'_k(j')|\rho_k=\rho\}|_{j\neq j'}f(\rho)d\rho \\ &= \frac{P}{v}\bar{\rho}\Pr\{\epsilon_{k,j}=\epsilon_{k,j'}=1, j\neq j'\}\end{aligned} \tag{13}$$

where $\mathrm{E}\{\rho_k\}$ is denoted by $\bar{\rho}$ for each k. But, $\Pr\{\epsilon_{k,j}=1\} = \frac{\binom{u-1}{v-1}}{\binom{u}{v}} = \frac{v}{u}$ and $\Pr\{\epsilon_{k,j}=\epsilon_{k,j'}=1, j\neq j'\} = \frac{\binom{u-2}{v-2}}{\binom{u}{v}} = \frac{v(v-1)}{u(u-1)}$. Let us define a $m\times m$ square matrix with all diagonal elements equal to a and all off-diagonal elements equal to b by $S(a,b;m)$. As such, $C_{k,i}$ can be expressed as:

$$C_{k,i} = \frac{|h_{k,i}|^2 P}{u} S(1, \bar{\rho}\frac{v-1}{u-1}; u). \tag{14}$$

Substituting this in (10), we get:

$$C(\vec{Z}_i) = \frac{P}{u} S(g_i, \bar{\rho} g_i \frac{v-1}{u-1}; u) + \sigma^2 I_u \tag{15}$$

where $g_i = \sum_{k\neq i} |h_{k,i}|^2$. We have $C(\vec{Y}_i(s_i^*)) = |h_{i,i}|^2 C(\vec{X}_i(s_i^*)) + C(\vec{Z}_i)$. It is clear that

$$C(\vec{X}_i(s_i^*)) = \frac{P}{v}\begin{pmatrix} S(1,\bar{\rho};v) & O_{v\times(u-v)} \\ O_{(u-v)\times v} & O_{(u-v)\times(u-v)} \end{pmatrix}. \tag{16}$$

Then:

$$C(\vec{Y}_i(s_i^*)) = \begin{pmatrix} \frac{|h_{i,i}|^2 P}{v} S(1,\bar{\rho};v) + \frac{P}{u} S(g_i, \bar{\rho} g_i \frac{v-1}{u-1}; v) + \sigma^2 I_v & \frac{P}{u}\bar{\rho} g_i \frac{v-1}{u-1} \mathbf{1}_{v,u-v} \\ \frac{P}{u}\bar{\rho} g_i \frac{v-1}{u-1} \mathbf{1}_{u-v,v} & \frac{P}{u} S(g_i, \bar{\rho} g_i \frac{v-1}{u-1}; u-v) + \sigma^2 I_{u-v} \end{pmatrix} \tag{17}$$

where we have shown a $a \times b$ matrix with all elements equal to one by $\mathbf{1}_{a,b}$. One may write more compactly

$$C(\vec{Z}_i) = S(t_{i,1}, t_{i,2}; u) \tag{18}$$

and

$$C(\vec{Y}_i) = \begin{pmatrix} S(t_{i,3}, t_{i,4}; v) & t_{i,2}\mathbf{1}_{v,u-v} \\ t_{i,2}\mathbf{1}_{u-v,v} & S(t_{i,1}, t_{i,2}; u-v) \end{pmatrix} \tag{19}$$

where $t_{i,1} = \frac{g_i P}{u} + \sigma^2$, $t_{i,2} = \frac{P}{u}\bar{\rho} g_i \frac{v-1}{u-1}$, $t_{i,3} = \frac{|h_{i,i}|^2 P}{v} + t_{i,1}$ and $t_{i,4} = \bar{\rho}\frac{|h_{i,i}|^2 P}{v} + t_{i,2}$.

To obtain the lower bound in (9), one has to compute det$(C(\vec{Z}_i))$ and det$(C(\vec{Y}_i(s_i^*)))$. The following lemma becomes handy in the sequel:

Lemma 1 *Let $a \neq b$ be real numbers. For any $S(a,b;m)$ the following hold:*

$$\det(S(a,b;m)) = (a-b)^m (1 + \frac{mb}{a-b})$$

$$S^{-1}(a,b;m) = \frac{1}{a-b}(I_m - \frac{b}{a+(m-1)b}\mathbf{1}_{m,1}\mathbf{1}_{m,1}^T).$$

Proof: We notice that for any two matrices $E_{m_1\times m_2}$ and $F_{m_2\times m_1}$, the following holds:

$$\det(I_{m_1} + EF) = \det(I_{m_2} + FE). \tag{*}$$

Also, for $A_{m_1\times m_1}$, $B_{m_1\times m_2}$, $C_{m_2\times m_2}$ and $D_{m_2\times m_1}$, we have the following result known as matrix inversion lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \tag{**}$$

One may write $S(a,b;m)$ as:

$$S(a,b;m) = (a-b)I_m + b\mathbf{1}_{m,1}\mathbf{1}_{m,1}^T.$$

Thus, based on (*), we get:

$$\det(S(a,b;m)) = (a-b)^m \det(I_m + \frac{b}{a-b}\mathbf{1}_{m,1}\mathbf{1}_{m,1}^T)$$

$$= (a-b)^m(1+\frac{b}{a-b}\mathbf{1}_{m,1}^T\mathbf{1}_{m,1}) = (a-b)^m(1+\frac{mb}{a-b}).$$

On the other hand, based on (**), we have:

$$S^{-1}(a,b;m) = \frac{1}{a-b}(I_m + \frac{b}{a-b}\mathbf{1}_{m,1}\mathbf{1}_{m,1}^T)^{-1} = \frac{1}{a-b}(I_m - \frac{b}{a+(m-1)b}\mathbf{1}_{m,1}\mathbf{1}_{m,1}^T).$$

■

According to this lemma, we get the following as a direct consequence:

$$\det(C(\vec{Z}_i)) = (t_{i,1}-t_{i,2})^u(1+\frac{ut_{i,2}}{t_{i,1}-t_{i,2}}). \tag{20}$$

To find $\det(C(\vec{Y}_i(s_i^*)))$, we invoke the following identity known as schur's lemma:

$$\det\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \det(A_1)\det(A_4 - A_3A_1^{-1}A_2) \tag{21}$$

where A_1, A_4 and the whole matrix are assumed to be square matrices. Applying this to the partitioned structure of $C(\vec{Y}_i(s_i^*))$, given in (17), yields the following:

$$\det(C(\vec{Y}_i(s_i^*))) = \det(S(t_{i,3},t_{i,4};v))\det(S(t_{i,1},t_{i,2};u-v) - t_{i,2}^2\mathbf{1}_{u-v,v}S^{-1}(t_{i,3},t_{i,4};v)\mathbf{1}_{v,u-v}). \tag{22}$$

Let us define $A = S(t_{i,1},t_{i,2};u-v) - t_{i,2}^2\mathbf{1}_{u-v,v}S^{-1}(t_{i,3},t_{i,4};v)\mathbf{1}_{v,u-v}$. According to the lemma, we have:

$$A = S(t_{i,1},t_{i,2};u-v) - \frac{t_{i,2}^2}{t_{i,3}-t_{i,4}}\mathbf{1}_{u-v,v}(I_v - \frac{t_{i,4}}{t_{i,3}+(v-1)t_{i,4}}\mathbf{1}_{v,1}\mathbf{1}_{v,1}^T)\mathbf{1}_{v,u-v}. \tag{23}$$

Since $\mathbf{1}_{u-v,v}\mathbf{1}_{v,u-v} = v\mathbf{1}_{u-v,u-v}$ and $\mathbf{1}_{u-v,v}\mathbf{1}_{v,1}\mathbf{1}_{v,1}^T\mathbf{1}_{v,u-v} = v^2\mathbf{1}_{u-v,u-v}$, this can be written as:

$$A = S(t_{i,1},t_{i,2};u-v) - \frac{t_{i,2}^2}{t_{i,3}-t_{i,4}}(v - \frac{v^2t_{i,4}}{t_{i,3}+(v-1)t_{i,4}})\mathbf{1}_{u-v,u-v}. \tag{24}$$

If we set $t_{i,5} = \frac{t_{i,2}^2}{t_{i,3}-t_{i,4}}(v-\frac{v^2t_{i,4}}{t_{i,3}+(v-1)t_{i,4}}) = \frac{vt_{i,2}^2}{t_{i,3}+(v-1)t_{i,4}}$, one has the following:

$$A = S(t_{i,1}-t_{i,5},t_{i,2}-t_{i,5};u-v). \tag{25}$$

Using this in (22), we have:

$$\det(C(\vec{Y}_i(s_i^*))) = \det(S(t_{i,3}, t_{i,4}; v))\det(S(t_{i,1} - t_{i,5}, t_{i,2} - t_{i,5}; u - v))$$

$$= (t_{i,3} - t_{i,4})^v (t_{i,1} - t_{i,2})^{u-v} (1 + \frac{vt_{i,4}}{t_{i,3} - t_{i,4}})(1 + \frac{(u - v)(t_{i,2} - t_{i,5})}{t_{i,1} - t_{i,2}}). \tag{26}$$

By (26), (20) and (9), we derive the following lower bound:

$$\mathscr{R}_i \geq L_i(v, f(.); \gamma) = \frac{1}{2} \log \left((\frac{t_{i,3} - t_{i,4}}{t_{i,1} - t_{i,2}})^v \frac{(1 + \frac{vt_{i,4}}{t_{i,3} - t_{i,4}})(1 + \frac{(u-v)(t_{i,2} - t_{i,5})}{t_{i,1} - t_{i,2}})}{1 + \frac{ut_{i,2}}{t_{i,1} - t_{i,2}}} \right)$$

$$= \frac{1}{2} \log \left((\frac{\tilde{t}_{i,3} - \tilde{t}_{i,4}}{\tilde{t}_{i,1} - \tilde{t}_{i,2}})^v \frac{(1 + \frac{v\tilde{t}_{i,4}}{\tilde{t}_{i,3} - \tilde{t}_{i,4}})(1 + \frac{(u-v)(\tilde{t}_{i,2} - \tilde{t}_{i,5})}{\tilde{t}_{i,1} - \tilde{t}_{i,2}})}{1 + \frac{u\tilde{t}_{i,2}}{\tilde{t}_{i,1} - \tilde{t}_{i,2}}} \right) \tag{27}$$

where $\tilde{t}_{i,j} = \frac{t_{i,j}}{\sigma^2}$.

IV. Upper Bounds on The Achievable Rates

To get an upper bound on $\mathscr{R}_i$, we proceed as follows. We start by finding an upper bound and a lower bound on $\mathrm{h}(\vec{Y}_i(s_i^*))$ and $\mathrm{h}(\vec{Z}_i)$ respectively. The former is simply derived if we replace $\vec{Y}_i(s_i^*)$ with a gaussian vector of the same covariance matrix. Therefore, we have:

$$\mathrm{h}(\vec{Y}_i(s_i^*)) \leq \frac{1}{2} \log((2\pi e)^u \det(C(\vec{Y}_i(s_i^*)))$$

$$= \frac{1}{2} u \log(2\pi e)$$

$$+ \frac{1}{2} \log \left((t_{i,3} - t_{i,4})^v (t_{i,1} - t_{i,2})^{u-v} (1 + \frac{vt_{i,4}}{t_{i,3} - t_{i,4}})(1 + \frac{(u - v)(t_{i,2} - t_{i,5})}{t_{i,1} - t_{i,2}}) \right)$$

$$= \frac{1}{2} u \log(2\pi e \sigma^2)$$

$$+ \frac{1}{2} \log \left((\tilde{t}_{i,3} - \tilde{t}_{i,4})^v (\tilde{t}_{i,1} - \tilde{t}_{i,2})^{u-v} (1 + \frac{v\tilde{t}_{i,4}}{\tilde{t}_{i,3} - \tilde{t}_{i,4}})(1 + \frac{(u - v)(\tilde{t}_{i,2} - \tilde{t}_{i,5})}{\tilde{t}_{i,1} - \tilde{t}_{i,2}}) \right). \tag{28}$$

Now, we focus to obtain an upper bound on $\mathrm{h}(\vec{Z}_i)$. Our strategy is based on using entropy power inequality repeatedly. Since the PDF of the random vector $\vec{\xi}_{k,i}$ is not smooth, no lower bound better than $-\infty$ is known for $\mathrm{h}(\vec{\xi}_{k,i})$. This results in a weak lower bound on $\mathrm{h}(\vec{Z}_i)$. To circumvent this, as $\vec{\eta}_i$ is a gaussian vector of covariance matrix equal to $\sigma^2 I_u$, we propose to decompose this random vector as the sum of $n-1$ independent Gaussian vectors of covariance matrices equal to $q_{k,i} I_u$. Denoting these gaussian vectors by $\vec{\eta}_{k,i}$, we perturb $\vec{\xi}_{k,i}$ by $\vec{\eta}_{k,i}$. The idea behind this perturbation is to smoothen the PDF of the vectors $\vec{\xi}_{k,i}$ so that entropy power inequality yields a tighter lower bound on $\mathrm{h}(\vec{Z}_i)$. Thus, one may write $\vec{Z}_i$ differently as follows:

$$\vec{Z}_i = \sum_{\substack{k=1\\k\neq i}}^{N} (h_{k,i}\vec{\xi}'_k + \vec{\eta}_{k,i}) \tag{29}$$

Defining $\vec{\nu}_{k,i} := h_{k,i}\vec{\xi}'_k + \vec{\eta}_{k,i}$, we have the following proposition

Proposition 1

$$\mathrm{h}(\vec{\nu}_{k,i}) \geq \frac{1}{2} u \log(2\pi e q_{k,i})$$
$$+\frac{1}{2}\int_0^1 \log\left((1+(1-\rho)\frac{|h_{k,i}|^2 P}{q_{k,i} v})^{v-1}(1+\frac{|h_{k,i}|^2 P}{q_{k,i}}(\rho+\frac{1-\rho}{v}))\right) f(\rho) d\rho.$$

Proof: See Appendix A. ■

As $\vec{Z}_i = \sum_{\substack{k=1\\k\neq i}}^{N} \vec{\nu}_{k,i}$ where $\{\vec{\nu}_{k,i}\}_{k\neq i}$ are independent, one may repeatedly use entropy power inequality to get a lower bound on $\mathrm{h}(\vec{Z}_i)$ as follows:

$$2^{\frac{2}{u}\mathrm{h}(\vec{Z}_i)} \geq \sum_{\substack{k=1\\k\neq i}}^{N} 2^{\frac{2}{u}\mathrm{h}(\vec{\nu}_{k,i})}. \tag{30}$$

Using proposition 2 in (30), one has the following lower bound on $\mathrm{h}(\vec{Z}_i)$:

$$\mathrm{h}(\vec{Z}_i) \geq \frac{u}{2}\log\left(2\pi e \sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i} 2^{\frac{1}{u}\int_0^1 \log\left((1+(1-\rho)\frac{|h_{k,i}|^2 P}{q_{k,i} v})^{v-1}(1+\frac{|h_{k,i}|^2 P}{q_{k,i}}(\rho+\frac{1-\rho}{v}))\right) f(\rho) d\rho}\right). \tag{31}$$

Since this is valid for any set of non-negative numbers $\{q_{k,i}\}_{k=1,k\neq i}^{N}$ satisfying $\sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i} = \sigma^2$, we tighten this lower bound as follows:

$$\mathrm{h}(\vec{Z}_i) \geq \frac{u}{2}\log(2\pi e \sigma_*^2(f(.))) \tag{32}$$

where

$$\sigma_*^2(f(.)) = \max_{q_{k,i}\geq 0:\sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i}=\sigma^2} \sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i} 2^{\frac{1}{u}\int_0^1 \log\left((1+(1-\rho)\frac{|h_{k,i}|^2 P}{q_{k,i} v})^{v-1}(1+\frac{|h_{k,i}|^2 P}{q_{k,i}}(\rho+\frac{1-\rho}{v}))\right) f(\rho) d\rho}. \tag{33}$$

The following lemma yields $\sigma_*^2(f(.))$:

Lemma 2

$$\sigma_*^2(f(.)) = 2^{\frac{1}{u}\int_0^1 \log\left((1+(1-\rho)\frac{g_i\gamma}{v})^{v-1}(1+g_i\gamma(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho}\sigma^2.$$

Proof:

To obtain $\sigma_*^2(f(.))$, let us define the Lagrangian as follows:

$$\mathcal{L} = \sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i} 2^{\frac{1}{u}\int_0^1 \log\left((1+(1-\rho)\frac{|h_{k,i}|^2 P}{q_{k,i} v})^{v-1}(1+\frac{|h_{k,i}|^2 P}{q_{k,i}}(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho} + \lambda(\sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i} - \sigma^2).$$

The optimality condition , $\frac{\partial L}{\partial q_{k,i}} = 0$, yields simply that the ratio $\frac{|h_{k,i}|^2}{q_{k,i}}$ must be a constant, namely ς, regardless of the value of k. Therefore, we get:

$$\sum_{\substack{k=1\\k\neq i}}^{N} q_{k,i} = \frac{1}{\varsigma}\sum_{\substack{k=1\\k\neq i}}^{N} |h_{k,i}|^2 = \sigma^2$$

which yields $\varsigma = \frac{g_i}{\sigma^2}$. As a result, the optimum value of $q_{k,i}$ is given by $q_{k,i} = \frac{|h_{k,i}|^2}{g_i}\sigma^2$. Consequently, $\sigma_*^2(f(.))$ is obtained as follows:

$$\sigma_*^2(f(.)) = 2^{\frac{1}{u}\int_0^1 \log\left((1+(1-\rho)\frac{g_i\gamma}{v})^{v-1}(1+g_i\gamma(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho}\sigma^2.$$

■

Substituting this in (32), we obtain the following lower bound on $h(\vec{Z}_i)$:

$$\mathrm{h}(\vec{Z}_i) \geq \frac{1}{2}u\log(2\pi e\sigma^2) + \frac{1}{2}\int_0^1 \log\left((1+(1-\rho)\frac{g_i\gamma}{v})^{v-1}(1+g_i\gamma(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho \tag{34}$$

By (28) and (34), we obtain the following upper bound on $\mathscr{R}_i$:

$$\mathscr{R}_i \leq U_i(v, f(.);\gamma)$$

$$:= \frac{1}{2}\log\left((\tilde{t}_{i,3}-\tilde{t}_{i,4})^v(\tilde{t}_{i,1}-\tilde{t}_{i,2})^{u-v}(1+\frac{v\tilde{t}_{i,4}}{\tilde{t}_{i,3}-\tilde{t}_{i,4}})(1+\frac{(u-v)(\tilde{t}_{i,2}-\tilde{t}_{i,5})}{\tilde{t}_{i,1}-\tilde{t}_{i,2}})\right)$$

$$-\frac{1}{2}\int_0^1 \log\left((1+(1-\rho)\frac{g_i\gamma}{v})^{v-1}(1+g_i\gamma(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho. \tag{35}$$

Denoting the sum-rate by SR, we come up with the following lower and upper bounds:

$$\sum_{i=1}^{n} L_i(v, f(.); \gamma) \leq SR \leq \sum_{i=1}^{n} U_i(v, f(.); \gamma). \tag{36}$$

Let us denote these lower and upper bounds by $L(v, f(.); \gamma)$ and $U(v, f(.); \gamma)$ respectively.

Before we proceed, we deem it appropriate to mention an issue. One could obtain a lower bound on $\mathscr{R}_i$ by following the same lines as we did to get an upper bound on $\mathscr{R}_i$. By vector perturbation and using entropy power inequality, one may get a lower bound on $\mathrm{h}(\vec{Y}_i(s_i^*))$, namely $\mathrm{h}_{lb}(\vec{Y}_i(s_i^*))$, and $\frac{1}{2}\log((2\pi e)^u \det(C(\vec{Z}_i)))$ would be an upper bound on $\mathrm{h}(\overrightarrow{Z}_i)$. Therefore, we come up with a new lower bound on $\mathscr{R}_i$ given by

$$\begin{aligned}\tilde{L}_i(v, f(.); \gamma) &= \mathrm{h}_{lb}(\vec{Y}_i(s_i^*)) - \frac{1}{2}\log((2\pi e)^u \det(C(\vec{Z}_i)))\\
&\leq \frac{1}{2}\log((2\pi e)^u \det(C(\vec{Y}_i(s_i^*)))) - \frac{1}{2}\log((2\pi e)^u \det(C(\vec{Z}_i)))\\
&= \frac{1}{2}\log\frac{\det(C(\vec{Y}_i))}{\det(C(\vec{Z}_i))} = L_i(x, f(\rho); \gamma)\end{aligned} \tag{37}$$

where the inequality is due to the fact that the Gaussian distribution maximizes the entropy of a random vector under a fixed covariance matrix condition. This shows that $L(v, f(.); \gamma)$ that we already found is a tighter lower bound than $\tilde{L}(v, f(.); \gamma)$.

V. Characterization of The Optimal Hopping Strategy

We start this section with the following key result.

Proposition 2 *Let $f(\rho)$ be any probability density function. Then $U_i(v, f(.); \gamma) \leq U_i(v, \delta(.); \gamma)$ for any $1 \leq i \leq N$.*

Proof: We give the proof in two steps.

Step 1 *According to (26), det$(C(\vec{Y}_i(s_i^*)))$ is given by:*

$$\mathit{det}(C(\vec{Y}_i(s_i^*))) = (t_{i,3} - t_{i,4})^v (t_{i,1} - t_{i,2})^{u-v} (1 + \frac{v t_{i,4}}{t_{i,3} - t_{i,4}})(1 + \frac{(u-v)(t_{i,2} - t_{i,5})}{t_{i,1} - t_{i,2}}).$$

We notice that $t_{i,3} \geq t_{i,4}$, $t_{i,1} \geq t_{i,2}$. Also, $t_{i,2}$ and $t_{i,4}$ are increasing linear functions in terms of $\bar{\rho}$, and

$t_{i,1}$ and $t_{i,3}$ are not functions of $\bar{\rho}$. On the other hand, $t_{i,5}$ vanishes as $\bar{\rho}=0$. As such, we have:

$$\frac{det(C(\vec{Y}_i(s_i^*)))}{det(C(\vec{Y}_i(s_i^*)))|_{f(.)=\delta(.)}} = \frac{(t_{i,3}-t_{i,4})^v(t_{i,1}-t_{i,2})^{u-v}(1+\frac{vt_{i,4}}{t_{i,3}-t_{i,4}})(1+\frac{(u-v)(t_{i,2}-t_{i,5})}{t_{i,1}-t_{i,2}})}{t_{i,3}^v t_{i,1}^{u-v}}$$

$$= (1-\frac{t_{i,4}}{t_{i,3}})^v(1-\frac{t_{i,2}}{t_{i,1}})^{u-v}(1+\frac{vt_{i,4}}{t_{i,3}-t_{i,4}})(1+\frac{(u-v)(t_{i,2}-t_{i,5})}{t_{i,1}-t_{i,2}})$$

$$\leq (1-\frac{t_{i,4}}{t_{i,3}})^v(1-\frac{t_{i,2}}{t_{i,1}})^{u-v}(1+\frac{vt_{i,4}}{t_{i,3}-t_{i,4}})(1+\frac{(u-v)t_{i,2}}{t_{i,1}-t_{i,2}}).$$

The inequality is valid as $t_{i,5}\geq 0$. Now, we verify that

$$(1-\frac{t_{i,4}}{t_{i,3}})^v(1+\frac{vt_{i,4}}{t_{i,3}-t_{i,4}}) \leq 1$$

and

$$(1-\frac{t_{i,2}}{t_{i,1}})^{u-v}(1+\frac{(u-v)t_{i,2}}{t_{i,1}-t_{i,2}}) \leq 1.$$

We prove the first claim. The proof of the second claim is exactly the same. Let us define $F(t_{i,4}) = (1-\frac{t_{i,4}}{t_{i,3}})^v(1+\frac{vt_{i,4}}{t_{i,3}-t_{i,4}})$. If $f(.)=\delta(.)$, then $t_{i,4}=0$ and $F(0)=1$. As $f(.)$ deviates from $\delta(.)$, $\bar{\rho}$ and therefore $t_{i,4}$ increases. To verify the claim, it suffices to show that $F(t_{i,4})$ is a decreasing function of $t_{i,4}$. One simply has $\frac{d}{dt_{i,4}}\ln F(t_{i,4}) = -\frac{v}{t_{i,3}-t_{i,4}}(1-\frac{t_{i,3}}{t_{i,3}+(v-1)t_{i,4}})$ which is negative, and we are done by the claims. As a result, we conclude the following:

$$\det(C(\vec{Y}_i(s_i^*))) \leq \det(C(\vec{Y}_i(s_i^*)))|_{f(.)=\delta(.)}.$$

Step 2 *Here, we show that $\sigma_*^2(\delta(.)) \leq \sigma_*^2(f(.))$ for any probability density function $f(.)$. By lemma 2, $\sigma_*^2(f(.)) = 2^{\frac{1}{u}\int_0^1 \log\left((1+(1-\rho)\frac{g_i\gamma}{v})^{v-1}(1+g_i\gamma(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho}\sigma^2$. Let us consider the function $G(\rho) = (1+(1-\rho)\frac{g_i\gamma}{v})^{v-1}(1+g_i\gamma(\rho+\frac{1-\rho}{v}))$. One simply has $\frac{d}{d\rho}\ln G(\rho) = -(g_i\gamma)^2(1-\frac{1}{v})\frac{\rho}{(1+g_i\gamma(\rho+\frac{1-\rho}{v}))(1+(1-\rho)\frac{g_i\gamma}{v})}$. This shows that $G(\rho)$ is a decreasing function of ρ. Thus:*

$$\sigma_*^2(f(.)) = 2^{\frac{1}{u}\int_0^1 \log(G(\rho))f(\rho)d\rho}\sigma^2 \geq 2^{\frac{1}{u}\log(G(0))\int_0^1 f(\rho)d\rho}\sigma^2$$

$$= 2^{\frac{1}{u}\log(G(0))\int_0^1 \delta(\rho)d\rho}\sigma^2 = 2^{\frac{1}{u}\int_0^1 \log(G(0))\delta(\rho)d\rho}\sigma^2$$

$$= 2^{\frac{1}{u}\int_0^1 \log(G(\rho))\delta(\rho)d\rho}\sigma^2 = \sigma_*^2(\delta(\rho)).$$

The claim of the proposition is clear now. As $U_i(v, f(.); \gamma) = \frac{1}{2}\log((2\pi e)^u \det(C(\vec{Y}_i(s_i^*)))) - \frac{1}{2}u\log(2\pi e\sigma_*^2(f(.)))$, based on the results of the above two steps, the claim of the proposition is proved. ■

From now on, we denote $U(v, \delta(.); \gamma)$ and $L(v, \delta(.); \gamma)$ by $U(v; \gamma)$ and $L(v; \gamma)$ respectively. Substituting $f(.) = \delta(.)$ in (27) and (35), we have:

$$L_i(v; \gamma) = \frac{1}{2}v\log\left(1 + \frac{|h_{i,i}|^2\gamma}{v(\frac{g_i\gamma}{u} + 1)}\right) \tag{38}$$

and

$$U_i(v; \gamma) = \frac{1}{2}v\log\left(1 + \frac{|h_{i,i}|^2\gamma}{v(\frac{g_i\gamma}{u} + 1)}\right) + \frac{1}{2}u\log(\frac{\gamma g_i}{u} + 1) - \frac{1}{2}v\log(1 + \frac{g_i\gamma}{v}). \tag{39}$$

Proposition 3 *For every realization of the crossover gains,*

$$\lim_{\gamma\to 0}\frac{L(u; \gamma)}{U(u-1; \gamma)} = 1.$$

Also, and as far as $\frac{\sum_{i=1}^N\left(\sum_{k=1, k\neq i}^N |h_{k,i}|^2\right)^2}{\sum_{i=1}^N |h_{i,i}|^4} < 1$,

$$\lim_{\gamma\to 0}\frac{d}{d\gamma}\frac{L(u; \gamma)}{U(u-1; \gamma)} > 0.$$

Proof: See Appendix B. ■

Proposition 4 *If* $\frac{\sum_{i=1}^N\left(\sum_{k=1, k\neq i}^N |h_{k,i}|^2\right)^2}{\sum_{i=1}^N |h_{i,i}|^4} < 1$*, there exists* $\gamma_0 > 0$ *such that for* $\gamma < \gamma_0$ *the function* $U(v; \gamma)$ *is an increasing function of* v.

Proof: See Appendix B. ■

Now, we are ready to express the main theorem of this section:

Theorem 1 *If* $\frac{\sum_{i=1}^N\left(\sum_{k=1, k\neq i}^N |h_{k,i}|^2\right)^2}{\sum_{i=1}^{i=N} |h_{i,i}|^4} < 1$*, then the best strategy for all users in terms of sum-rate maximization is to set* $f(.) = \delta(.)$ *and to spread their power on the whole available band, i.e.,* $v = u$.

Proof: By proposition 4, there exists a $\gamma_1 > 0$ such that if $\gamma < \gamma_1$ then for $\frac{\sum_{i=1}^N\left(\sum_{k=1, k\neq i}^N |h_{k,i}|^2\right)^2}{\sum_{i=1}^N |h_{i,i}|^4} < 1$ we have $U(u-1; \gamma) < L(u; \gamma)$. On the other hand, by proposition 5, there exists a $\gamma_0 > 0$ such that if $\gamma < \gamma_0$ then for $\frac{\sum_{i=1}^N\left(\sum_{k=1, k\neq i}^N |h_{k,i}|^2\right)^2}{\sum_{i=1}^N |h_{i,i}|^4} < 1$ we have $U(u-1; \gamma) > U(t; \gamma)$ where $t \in \{1, 2, ..., u-2\}$. As such, taking $\frac{\sum_{i=1}^N\left(\sum_{k=1, k\neq i}^N |h_{k,i}|^2\right)^2}{\sum_{i=1}^N |h_{i,i}|^4} < 1$ and for every $\gamma < \min\{\gamma_0, \gamma_1\}$, we have $L(u, \gamma) > U(t; \gamma)$ where $t \in \{1, 2, ..., u-$

$1\}$. Also, as proved in proposition 3, $U(t;\gamma) \geq U(t, f(\rho);\gamma)$ for any distribution $f(\rho)$. Therefore, we conclude that n the low SNR regime taking $v = u$ and $f(.) = \delta(.)$ yields a higher SR than $v < u$ and any arbitrary PDF $f(.)$. One can easily check that $L(u;\gamma) = U(u;\gamma) \geq U(u, f(.);\gamma)$. Summarizing the above, we see that SR is maximized for $v = u$ and $f(.) = \delta(.)$ as long as $\frac{\sum_{i=1}^{N}\left(\sum_{k=1,k\neq i}^{N}|h_{k,i}|^2\right)^2}{\sum_{i=1}^{N}|h_{i,i}|^4} < 1$. ■

It is notable that in a decentralized network, different users are not necessarily aware of all the channel gains. Theorem 3, offers a criterion which requires all the users to be aware of $h_{i,i}$ and g_i for all i. This might not be applicable in a distributed network. On the other hand, the users might be able to bound these quantities. Assume that it is almost surely true that $|h_{i,i}| > \epsilon_1$ and $|h_{i,j}| < \epsilon_2$ for $i \neq j$ where ϵ_1 and ϵ_2 are specific thresholds. Then, $\frac{\sum_{i=1}^{N}\left(\sum_{\substack{k=1\\k\neq i}}^{N}|h_{k,i}|^2\right)^2}{\sum_{i=1}^{N}|h_{i,i}|^4} < \frac{(N-1)^2\epsilon_2^4}{\epsilon_1^4}$. Therefore, $\frac{\epsilon_2}{\epsilon_1} < \frac{1}{\sqrt{N-1}}$ is a sufficient condition for all the users to distribute their power on the whole band in the low SNR regime. For example, if $N = 2$, then $\epsilon_2 < \epsilon_1$, i.e., the crossover gains be smaller than the forward gains. We are able to give a more detailed argument in the special case $N = 2$ in terms of offering a computable low SNR range. Let us call the users as A and B. We suppose the forward gains are one and the crossover gains of user A on user B and user B on user A are a and b respectively. We suppose $a, b < 1$. By the theorem above, we know that in the low SNR regime, the best choice would be to occupy all the available band. We show that as long as $\frac{\gamma}{u} < \frac{\sqrt{5}-1}{4}$ the same conclusion holds, and as such, $[0, \frac{\sqrt{5}-1}{4}u]$ is an explicit characterization of the low SNR regime. For the moment, let us assume that $a = b = 1$. Let link A, occupy the first v bands. The other transmitter also uses v bands of which a number of v^* bands are among the first v bands. Clearly, we have $v^* \leq v$ and $v - v^* \leq u - v$ which yields $\max\{2v - u, 0\} \leq v^* \leq v$. In this case, it is easy to check that the achievable rate of user A is:

$$R_A(v^*) = \frac{1}{2}\log\left((1+\frac{2P}{v\sigma^2})^{v^*}(1+\frac{P}{v\sigma^2})^{v-2v^*}\right) = \frac{1}{2}v^*\log(1+\frac{2\gamma}{v}) + \frac{1}{2}(v-2v^*)\log(1+\frac{\gamma}{v}) \quad (40)$$

On the other hand, for a fixed input distribution, the mutual information for an additive noise channel is a convex function of the noise PDF. Thus, we obtain the following:

$$R_A \leq \sum_{v^*=\max\{0,2v-u\}}^{v} p_{v^*} R_A(v^*) \quad (41)$$

where p_{v^*} is the probability that the two users coincide on v^* sub-bands. Clearly, $p_{v^*} = \frac{\binom{v}{v^*}\binom{u-v}{v-v^*}}{\binom{u}{v}}$. Denoting the above upper bound by UB, we get:

$$UB = \frac{1}{2}\left(\sum_{v^*=\max\{2v-u,0\}}^{v} y\frac{\binom{v}{v^*}\binom{u-v}{v-v^*}}{\binom{u}{v}}\right)\log(1+\frac{2\gamma}{v})$$

$$+\frac{1}{2}\left(\sum_{v^*=\max\{2v-u,0\}}^{v} (v-2v^*)\frac{\binom{v}{v^*}\binom{u-v}{v-v^*}}{\binom{u}{v}}\right)\log(1+\frac{\gamma}{v}). \tag{42}$$

We recall that the probability function of a hypergeometric random variable T is given by:

$$\Pr\{T=t\} = \frac{\binom{M_1}{t}\binom{M_2-M_1}{m-t}}{\binom{M_2}{m}} \tag{43}$$

where $\max\{0, M_1+m-M_2\} \leq t \leq \min\{M_1, m\}$. Also, one has $E\{T\} = \frac{M_1 m}{M_2}$. If we set $M_1 = m = v$ and $M_2 = u$, then we see that $\frac{\binom{v}{v^*}\binom{u-v}{v-v^*}}{\binom{u}{v}}$ is actually a hypergeometric probability function. As such, the summation terms in (42) are computed as follows:

$$\sum_{v^*=\max\{2v-u,0\}}^{v} v^*\frac{\binom{v}{v^*}\binom{u-v}{v-v^*}}{\binom{u}{v}} = \frac{v^2}{u} \tag{44}$$

and

$$\sum_{v^*=\max\{2v-u,0\}}^{v} \frac{\binom{v}{v^*}\binom{u-v}{v-v^*}}{\binom{u}{v}} = 1 \tag{45}$$

Replacing these terms in (42), we get:

$$UB = \frac{1}{2}\frac{v^2}{u}\log(1+\frac{2\gamma}{v}) + \frac{1}{2}(v - \frac{2v^2}{u})\log(1+\frac{\gamma}{v}). \tag{46}$$

It is interesting to note that $UB|_{v=u} = R_A(u)$,i.e, the upper bound is tight at $v = u$. We just need to see for which range of SNR the upper bound is an increasing function of v. In fact, if UB is an increasing function of v, the optimum value of v to maximize SR would be u. We have the following proposition:

Proposition 5 *UB is an increasing function of v as long as $\gamma \in [0, \frac{\sqrt{5}-1}{4}u]$.*

Proof: See Appendix C. ■

Hence, $[0, \frac{\sqrt{5}-1}{4}u]$ is an explicit range of SNR for which sum-rate is maximized if $v = u$ for all $a, b < 1$. We notice that by Theorem 3, for all a, b satisfying $a^4 + b^4 \leq 2$, the optimum choice is $v = u$. In this example, we are assuming that $a, b \in [0, 1]$ which is included in the region specified by Theorem 3. The following figure illustrates the regions specified here.

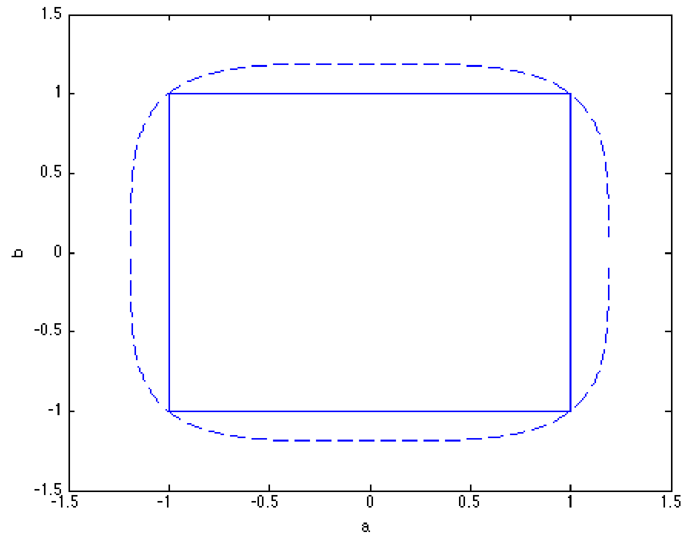

Fig. 1. Dash Line: The region $a^4 + b^4 \leq 2$, Straight Line: The region $0 \leq a, b \leq 1$

remark 1 *It is easy to see that in general for $N = 2$,*

$$\mathscr{R}_i \leq (v - \frac{v^2}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v}) + \frac{v^2}{u} \log\left(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v}}{1 + \frac{|h_{i',i}|^2\gamma}{v}}\right) \tag{47}$$

where for $i \in \{1, 2\}$, $i' := 3 - i$.

Now, we consider a setup in a decentralized network of two users where the number of active users and the channel gains are unknown to all transmitters. We set $q_n = \Pr\{N = n\}$ for $1 \leq n \leq 2$. Hence, the randomness of the number of users contributes in the outage event. Denoting this event for the i^{th} user by $\mathcal{O}_i$, we have:

$$\mathcal{O}_i = \{N, h_{1,i}, h_{2,i} : \mathscr{R}_i < R\} \tag{48}$$

where R is the transmission rate of the i^{th} user. We define the $\epsilon-$outage capacity of any user with hopping parameter v by:

$$R(\epsilon; v) := \sup\{R : \Pr\{\mathcal{O}_i\} \leq \epsilon\}. \tag{49}$$

We aim to show that for low SNR values, $R(\epsilon; v)$ is maximized at $v = u$.

Let

$$R_{ub}(\epsilon; v) := \sup\{R : \Pr\{N, h_{1,i}, h_{2,i} : \mathscr{R}_{i,ub} < R\} \leq \epsilon\} \tag{50}$$

where

$$\mathscr{R}_{i,ub} := \begin{cases} v\log(1+\frac{|h_{i,i}|^2\gamma}{v}) & N=1 \\ (v-\frac{v^2}{u})\log(1+\frac{|h_{i,i}|^2\gamma}{v}) + \frac{v^2}{u}\log\left(1+\frac{\frac{|h_{i,i}|^2\gamma}{v}}{1+\frac{|h_{i',i}|^2\gamma}{v}}\right) & N=2 \end{cases}. \tag{51}$$

$\mathscr{R}_{i,ub}$ is an upper bound on $\mathscr{R}_i$ in case $N=1$ and $N=2$ respectively. Clearly[2], $\{\mathscr{R}_{i,ub} < R\} \subset \mathcal{O}_i$. This yields

$$\{R : \Pr\{\mathcal{O}_i\} \leq \epsilon\} \subset \{R : \Pr\{\mathscr{R}_{i,ub} < R\} \leq \epsilon\}. \tag{52}$$

Thus,

$$R(\epsilon; v) \leq R_{ub}(\epsilon; v). \tag{53}$$

Proposition 6 *If $v < u$,*

$$R_{ub}(\epsilon; v)$$

$$= \sup\{R : q_1(1-\exp\left(\frac{v}{\gamma}(1-2^{\frac{R}{v}})\right)+q_2\int_0^\infty \left(1_{\mathcal{B}}+\exp\left(\frac{v}{\gamma}-\frac{z}{2^{\frac{uR}{v^2}}(1+\frac{z\gamma}{v})^{1-\frac{u}{v}}-1}\right)(1_{\mathcal{A}}-1_{\mathcal{B}})\right)\exp(-z)dz \leq \epsilon\} \tag{54}$$

where

$$\mathcal{A} = \{z : 2^{\frac{uR}{v^2}}(1+\frac{z\gamma}{v})^{1-\frac{u}{v}} > 1\} \tag{55}$$

and

$$\mathcal{B} = \{z : 2^{\frac{uR}{v^2}}(1+\frac{z\gamma}{v})^{-\frac{u}{v}} > 1\}. \tag{56}$$

Also, if $v = u$,

$$R_{ub}(\epsilon; u) = \sup\{R : 1-\exp(\frac{u}{\gamma}(1-2^{\frac{R}{u}}))(q_1+q_2 2^{-\frac{R}{u}}) \leq \epsilon\}. \tag{57}$$

Proof: See appendix D. ■

Fig. 3 sketches $R_{ub}(\epsilon; v)$ for $1 \leq v \leq 4$ in a system with $u = 4$ at $\gamma = 0$dB. It is seen that all the curves overlap on each other implying that hopping has no particular advantage. It is notable that $R_{ub}(\epsilon; v=4)$ is the exact ϵ-outage capacity as $\mathscr{R}_{i,ub}$ is tight for $v = u$. Therefore, we conclude that ϵ-outage capacity for $v = 4$ is at least as large as the same quantity in case $v < 4$.

[2]By $\{\mathscr{R}_{i,ub} < R\}$, we mean $\{N, h_{1,i}, h_{2,i} : \mathscr{R}_{i,ub} < R\}$.

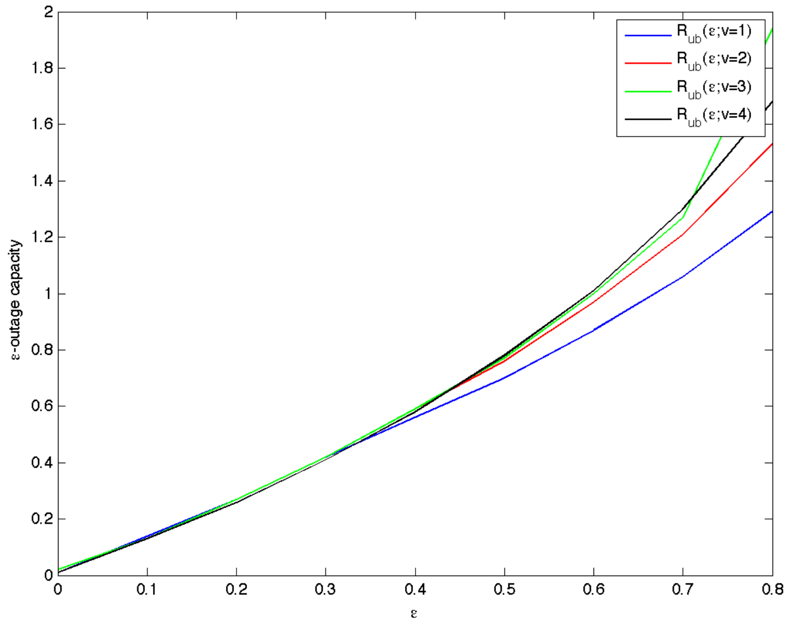

Fig. 2. Depiction of $R_{ub}(\epsilon; v)$ for $1 \leq v \leq 4$ in a system with $u = 4$ at $\gamma = 0$dB

VI. APPENDIX A

In this appendix, we prove proposition 1. We are concerned to get a lower bound on the differential entropy of $\vec{\nu}_{k,i} := h_{k,i}\vec{\xi}'_k + \vec{\eta}_{k,i}$. If we set $\vec{\tau}_{k,i} := h_{k,i}\vec{\xi}'_k$, then we have $p_{\vec{\tau}_{k,i}}(\vec{\tau}) = \frac{1}{\binom{u}{v}} \sum_{l=1}^{\binom{u}{v}} \int_0^1 g(\vec{\tau}, D_{l,k,i}) f(\rho) d\rho$ where each $D_{l,k,i}$ is a matrix which has a $v \times v$ principal sub-matrix equal to $\frac{|h_{k,i}|^2 P}{v} S(1, \rho; v)$ and the rest of its elements are zero. Each $D_{l,k,i}$ shows a specific occupation of v frequency bands out of the u bands. We have

$$p_{\vec{\nu}_{k,i}}(\vec{\nu}) = p_{\vec{\tau}_{k,i}}(\vec{\nu}) * g(\vec{\nu}, q_{k,i} I_u)$$

$$= \frac{1}{\binom{u}{x}} \sum_{l=1}^{\binom{u}{v}} \int_0^1 (g(\vec{\nu}, D_{l,k,i}) * g(\vec{\nu}, q_{k,i} I_u)) f(\rho) d\rho = \frac{1}{\binom{u}{v}} \sum_{i=1}^{\binom{u}{v}} \int_0^1 g(\vec{\nu}, D_{l,k,i} + q_{k,i} I_u) f(\rho) d\rho.$$

Since differential entropy is a concave function of probability density function, we get the following result:

$$\text{h}(\vec{\nu}_{k,i}) \geq \frac{1}{2} \frac{1}{\binom{u}{v}} \sum_{l=1}^{\binom{u}{v}} \int_0^1 \log((2\pi e)^u \det(D_{l,k,i} + q_{k,i} I_u)) f(\rho) d\rho. \quad (*)$$

Clearly, $\det(D_{l,k,i} + q_{k,i} I_u)$ is independent of l. To compute this quantity, we consider the case where the first v rows and v columns of $D_{l,k,i}$ consist the aforementioned principal sub-matrix which is equal to

$\frac{P}{v}S(1,\rho;v)$. In this, we have:

$$\det(D_{l,k,i}+q_{k,i}I_u)=\det\begin{pmatrix} S(\frac{|h_{k,i}|^2P}{v}+q_{k,i},\frac{|h_{k,i}|^2P}{v}\rho;v) & \mathbf{0}_{v,u-v} \\ \mathbf{0}_{u-v,v} & q_{k,i}I_{u-v}\end{pmatrix}$$

$$=q_{k,i}^{u-v}\det(S(\frac{|h_{k,i}|^2P}{v}+q_{k,i},\frac{|h_{k,i}|^2P}{v}\rho;v))=q_{k,i}^{u-v}(\frac{|h_{k,i}|^2P}{v}(1-\rho)+q_{k,i})^v(1+\frac{\rho|h_{k,i}|^2}{\frac{|h_{k,i}|^2P}{v}(1-\rho)+q_{k,i}})$$

$$=q_{k,i}^{u}(\frac{|h_{k,i}|^2P}{vq_{k,i}}(1-\rho)+1)^v(1+\frac{\frac{\rho|h_{k,i}|^2P}{q_{k,i}}}{\frac{|h_{k,i}|^2P}{vq_{k,i}}(1-\rho)+1}).$$

Substituting this in (*), we get:

$$\mathrm{h}(\vec{\nu}_{k,i})\geq\frac{1}{2}u\log(2\pi e q_{k,i})+\frac{1}{2}\int_0^1\log\left((1+(1-\rho)\frac{|h_{k,i}|^2P}{q_{k,i}v})^v(1+\frac{\frac{\rho|h_{k,i}|^2P}{q_{k,i}}}{\frac{|h_{k,i}|^2P}{vq_{k,i}}(1-\rho)+1}))\right)f(\rho)d\rho$$

$$=\frac{1}{2}u\log(2\pi e q_{k,i})+\frac{1}{2}\int_0^1\log\left((1+(1-\rho)\frac{|h_{k,i}|^2P}{q_{k,i}v})^{v-1}(1+\frac{|h_{k,i}|^2P}{q_{k,i}}(\rho+\frac{1-\rho}{v}))\right)f(\rho)d\rho$$

which is the desired result.

VII. Appendix B

Let us define $g_i=\sum_{\substack{k=1\\k\neq i}}^N|h_{k,i}|^2$ and $f_i=|h_{i,i}|^2$. We notice that the following holds:

$$U(v;\gamma)=L(v;\gamma)+\Delta(v;\gamma)$$

where $\Delta(v;\gamma)=\frac{1}{2}u\sum_{i=1}^N\log(\frac{\gamma g_i}{u}+1)-\frac{1}{2}v\sum_{i=1}^N\log(1+\frac{g_i\gamma}{v})$. As $L(u;0)=U(u-1;0)=0$, we have:

$$\lim_{\gamma\to0}\frac{L(u;\gamma)}{U(u-1;\gamma)}=\frac{L'(u;0)}{U'(u-1;0)}$$

and

$$\lim_{\gamma\to0}\frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{L(u;\gamma)}{U(u-1;\gamma)}=\frac{U'(u-1;0)L''(u;0)-L'(u;0)U''(u-1;0)}{2(U'(u-1;0))^2}.$$

After simple calculations, we get:

$$L'(v,0)=U'(v;0)=\frac{\sum_i f_i}{2}, \quad (*)$$

$$L''(v;0)=-\frac{1}{2}\sum_i f_i(\frac{f_i}{v}+\frac{2g_i}{u}),$$

$$\Delta''(v;0) = \frac{1}{2}\sum_i g_i^2(\frac{1}{v} - \frac{1}{u})$$

and

$$U''(v;0) = L''(v;0) + \Delta''(v;0) = -\frac{1}{2}\sum_i f_i(\frac{f_i}{v} + \frac{2g_i}{u}) + \frac{1}{2}\sum_i g_i^2(\frac{1}{x} - \frac{1}{u}). \qquad (**)$$

As $L'(u;0) = U'(u-1;0)$ the first part of proposition 4 is derived. By the same token, the second part is deduced if the condition $L''(u;0) > U''(u-1;0)$ is satisfied. This yields the following:

$$\sum_i g_i^2(\frac{1}{u-1} - \frac{1}{u}) < \sum_i f_i^2(\frac{1}{u-1} - \frac{1}{u})$$

which is simplified to $\frac{\sum_i g_i^2}{\sum_i f_i^2} < 1$. To prove proposition 5, we show the following two claims hold for $s > t$:

$$\lim_{\gamma \to 0} \frac{U(s;\gamma)}{U(t;\gamma)} = 1$$

and

$$\lim_{\gamma \to 0} \frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{U(s;\gamma)}{U(t;\gamma)} > 0.$$

as $U(s;0) = U(t;0) = 0$, we have:

$$\lim_{\gamma \to 0} \frac{U(s;\gamma)}{U(t;\gamma)} = \frac{U'(s;0)}{U'(t,0)}$$

and

$$\lim_{\gamma \to 0} \frac{\mathrm{d}}{\mathrm{d}\gamma}\frac{U(s;\gamma)}{U(t;\gamma)} = \frac{U'(t;0)L''(s;0) - L'(s;0)U''(t;0)}{2(U'(t;0))^2}.$$

By (*), we have:

$$U'(s;0) = U'(t,0) = \frac{\sum_i f_i}{2}$$

which yields the first claim. Considering this fact, the second claim is derived whenever $U''(s;0) > U''(t;0)$. According to (**), this can be written as:

$$\sum_i g_i^2 < \sum_i f_i^2$$

which yields $\frac{\sum_i g_i^2}{\sum_i f_i^2} < 1$.

VIII. APPENDIX C

We have:

$$UB = \frac{v^2}{u} \log \frac{\sqrt{1+\frac{2\gamma}{v}}}{1+\frac{\gamma}{v}} + \frac{1}{2} \log(1+\frac{\gamma}{v})$$

$$= u \left(\frac{v^2}{u^2} \log \frac{\sqrt{1+\frac{2\gamma}{v}}}{1+\frac{\gamma}{v}} + \frac{1}{2} \frac{v}{u} \log(1+\frac{\gamma}{v}) \right).$$

Let $c = \frac{\gamma}{u}$ and $w = \frac{v}{u}$. Therefore,

$$UB = u \left(w^2 \log \frac{\sqrt{1+\frac{2c}{w}}}{1+\frac{c}{w}} + \frac{1}{2} w \log(1+\frac{c}{w}) \right)$$

$$= u \left(w^2 \log \frac{\sqrt{w(w+2c)}}{w+c} + \frac{1}{2} w \log \frac{w+c}{w} \right)$$

$$= u \left(\frac{w^2-w}{2} \log w + \frac{w^2}{2} \log(w+2c) - (w^2 - \frac{w}{2}) \log(w+c) \right).$$

Let us define:

$$\varphi(w,c) = \frac{w^2-w}{2} \ln w + \frac{w^2}{2} \ln(w+2c) - (w^2 - \frac{w}{2}) \ln(w+c).$$

We have:

$$\frac{\partial \varphi(w,c)}{\partial w} = \frac{w^2-w}{2} \frac{1}{w} + \frac{w^2}{2} \frac{1}{w+2c} - (w^2 - \frac{w}{2}) \frac{1}{w+c}$$

$$+ \frac{2w-1}{2} \ln w + w \ln(w+2c) - (2w - \frac{1}{2}) \ln(w+c).$$

One observes that $\forall w \in (0,1] : \frac{\partial \varphi(w,c)}{\partial v}|_{c=0} = 0$. On the other hand, $\frac{\partial^2 \varphi(w,c)}{\partial c \partial w}$ is computed as follows:

$$\frac{\partial^2 \varphi(w,c)}{\partial c \partial w} = \frac{c\,(4(1-2w)c^2 + 2w(2-3w)c + w^2))}{2(w+c)^2(w+2c)^2}.$$

Now, for each value of $w \in (0,1]$, we investigate the behavior of the following quadratic polynomial as a function of c:

$$\psi_w(c) = 4(1-2w)c^2 + 2w(2-3w)c + w^2.$$

The following cases occur:

- $w = \frac{1}{2}$

In this case, $\psi_{\frac{1}{2}}(c) = \frac{1}{2}c + \frac{1}{4}$ is a line which is positive for all $c \geq 0$.

- $w \in (0, \frac{1}{2})$

In this case, $\psi_w(c)$ is a parabola that has a minimum at $c_0 = -\frac{w(2-3w)}{4(1-2w)}$. Clearly, $c_0 < 0$ for $w \in (0, \frac{1}{2})$. On the other hand, $\psi_w(0) = w^2 > 0$. Hence, $\forall c \geq 0 : \psi_w(c) \geq 0$.

- $w \in (\frac{1}{2}, 1]$

In this case, $\psi_w(c)$ is a parabola achieving its maximum at c_0 which is a positive number for $w \in (\frac{1}{2}, \frac{2}{3})$ and a negative one for $w \in (\frac{2}{3}, 1]$. On the other hand, the roots of $\psi_w(c)$ are given by:

$$c_{1,2} = \frac{-w(2-3w) \pm w\sqrt{w(9w-4)}}{4(1-2w)}.$$

The term $(9w - 4)$ is positive for $w \in (\frac{1}{2}, 1]$, and therefore the root are real. Since $\psi_w(0) = w^2$, one of the real roots is always positive and the other one is always negative. Denoting the positive root by c_+, we have:

$$c_+ = \frac{-w(2-3w) - w\sqrt{w(9w-4)}}{4(1-2w)}.$$

Let us sketch c_+ as a function of $w \in (\frac{1}{2}, 1]$. As can be seen from this figure, c_+ is a monotonically decreasing function of w. As such, we have $\inf_{w \in (\frac{1}{2}, 1]} c_+ = c_+|_{w=1} = \frac{\sqrt{5}-1}{4}$. From the above, we conclude:

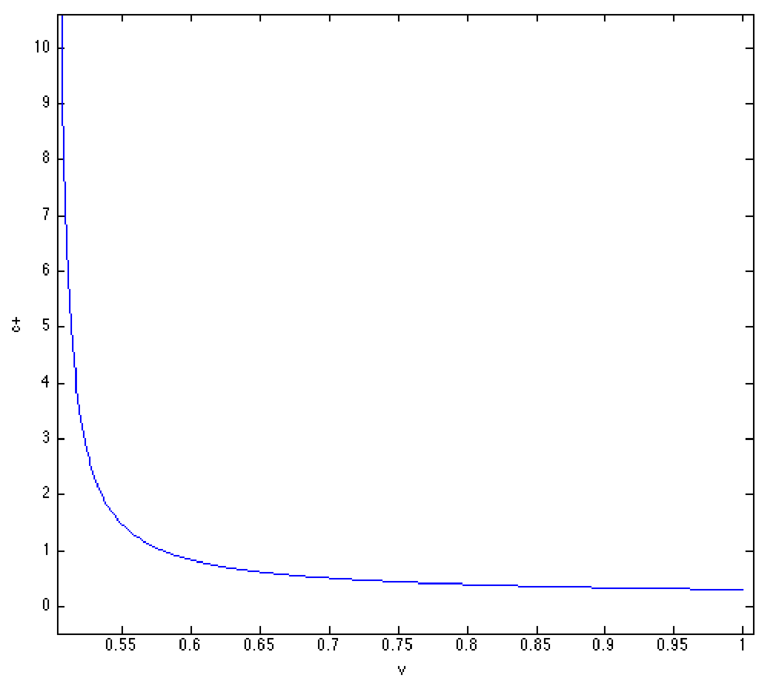

Fig. 3. $c_+(w)$ is a decreasing function of v for $v \in (\frac{1}{2}, 1)$

$$\forall w \in (0,1], \forall c \in [0, \frac{\sqrt{5}-1}{4}] : \frac{\partial^2 \varphi(w,c)}{\partial c \partial w} > 0.$$

As $\forall w \in (0,1] : \frac{\partial \varphi(w,c)}{\partial v}|_{c=0} = 0$, this has a nice interpretation. For each $c \in [0, \frac{\sqrt{5}-1}{4}]$, $\varphi(w,c)$ is an increasing function of w, and the theorem is proved.

IX. Appendix D

In what follows, we derive an expression for $R_{ub}(\epsilon; v)$. We have:

$$\Pr\{\mathscr{R}_{i,ub} < R\} = q_1 \Pr\{\mathscr{R}_{i,ub} < R|N=1\} + q_2 \Pr\{\mathscr{R}_{i,ub} < R|N=2\}. \tag{58}$$

Taking $h_{1,i}$ and $h_{2,i}$ to be $\mathcal{CN}(0,1)$, then $|h_{1,i}|^2$ and $|h_{2,i}|^2$ are exponential random variables of parameter one. It is easy to see that

$$\Pr\{\mathscr{R}_{i,ub} < R|N=1\} = 1 - \exp(\frac{v}{\gamma}(1 - 2^{\frac{R}{v}})). \tag{59}$$

On the other hand, in case $v < u$,

$$\Pr\{\mathscr{R}_{i,ub} < R|N=2\} = \Pr\{(v - \frac{v^2}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v}) + \frac{v^2}{u} \log\left(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v}}{1 + \frac{|h_{i',i}|^2\gamma}{v}}\right) < R\}$$

$$= E_{h_{i,i}}\Big\{ \Pr\left\{(v - \frac{v^2}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v}) + \frac{v^2}{u} \log\left(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v}}{1 + \frac{|h_{i',i}|^2\gamma}{v}}\right) < R\Big| h_{i,i}\right\}\Big\}. \tag{60}$$

But,

$$\Pr\left\{(v - \frac{v^2}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v}) + \frac{v^2}{u} \log\left(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v}}{1 + \frac{|h_{i',i}|^2\gamma}{v}}\right) < R\Big| h_{i,i}\right\}$$

$$= \mu_1(\mu_2 + \exp\left(\frac{v}{\gamma} - \frac{|h_{i,i}|^2}{2^{\frac{uR}{v^2}}(1 + \frac{|h_{i,i}|^2\gamma}{v})^{1-\frac{u}{v}} - 1}\right)\bar{\mu}_2) \tag{61}$$

where

$$\mu_1 = \begin{cases} 1 & 2^{\frac{uR}{v^2}}(1 + \frac{|h_{i,i}|^2}{v})^{1-\frac{u}{v}} > 1 \\ 0 & \text{oth.} \end{cases} \tag{62}$$

and

$$\mu_2 = \begin{cases} 1 & 2^{\frac{uR}{v^2}}(1 + \frac{|h_{i,i}|^2}{v})^{-\frac{u}{v}} > 1 \\ 0 & \text{oth.} \end{cases} . \tag{63}$$

Clearly, if $\mu_2 = 1$, then $\mu_1 = 1$. Thus $\mu_1\mu_2 = \mu_2$. Hence,

$$\Pr\left\{(v - \frac{v^2}{u})\log(1 + \frac{|h_{i,i}|^2\gamma}{v}) + \frac{v^2}{u}\log\left(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v}}{1 + \frac{|h_{i',i}|^2\gamma}{v}}\right) < R\middle|h_{i,i}\right\}$$

$$= \mu_2 + \exp\left(\frac{v}{\gamma} - \frac{|h_{i,i}|^2}{2^{\frac{uR}{v^2}}(1 + \frac{|h_{i,i}|^2\gamma}{v})^{1-\frac{u}{v}} - 1}\right)(\mu_1 - \mu_2). \tag{64}$$

Finally, we have:

$$\Pr\{\mathscr{R}_{i,ub} < R | N = 2\}$$

$$= \int_0^\infty \left(1_{\mathcal{B}} + \exp\left(\frac{v}{\gamma} - \frac{z}{2^{\frac{uR}{v^2}}(1 + \frac{z\gamma}{v})^{1-\frac{u}{v}} - 1}\right)(1_{\mathcal{A}} - 1_{\mathcal{B}})\right)\exp(-z)dz \tag{65}$$

where

$$\mathcal{A} = \{z : 2^{\frac{uR}{v^2}}(1 + \frac{z\gamma}{v})^{1-\frac{u}{v}} > 1\} \tag{66}$$

and

$$\mathcal{B} = \{z : 2^{\frac{uR}{v^2}}(1 + \frac{z\gamma}{v})^{-\frac{u}{v}} > 1\}. \tag{67}$$

Therefore,

$$R_{ub}(\epsilon; v)$$

$$= \sup\{R : q_1(1 - \exp\left(\frac{v}{\gamma}(1 - 2^{\frac{R}{v}})\right) + q_2\int_0^\infty \left(1_{\mathcal{B}} + \exp\left(\frac{v}{\gamma} - \frac{z}{2^{\frac{uR}{v^2}}(1 + \frac{z\gamma}{v})^{1-\frac{u}{v}} - 1}\right)(1_{\mathcal{A}} - 1_{\mathcal{B}})\right)\exp(-z)dz \leq \epsilon\}. \tag{68}$$

If $v = u$, after similar calculations, we get:

$$R_{ub}(\epsilon; u) = \sup\{R : 1 - \exp(\frac{u}{\gamma}(1 - 2^{\frac{R}{u}}))(q_1 + q_2 2^{-\frac{R}{u}}) \leq \epsilon\}. \tag{69}$$

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