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Spectrum Sharing vs Spectral Efficiency in Decentralized Networks

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Abstract

We consider a wireless communication network with a fixed number of frequency sub-bands to be shared among several transmitter-receiver pairs. In traditional frequency division (FD) systems, the available sub-bands are partitioned into disjoint clusters (frequency bands) and assigned to different users (each user transmits only in its own band). If the number of users sharing the spectrum is random, this technique may lead to inefficient spectrum utilization (a considerable fraction of the bands may remain empty most of the time). In addition, this approach inherently requires either a central network controller for frequency allocation, or cognitive radios which sense and occupy the empty bands in a dynamic fashion. These shortcomings motivate us to look for a decentralized scheme (without using cognitive radios) which allows the users to coexist, while utilizing the spectrum efficiently. We consider a frequency hopping (FH) scheme (with iid Gaussian code-books) where each user transmits over a selection of sub-bands and hops to another selection (with the same cardinality) from transmission to transmission. We derive lower and upper bounds on the achievable rate of each user and demonstrate that for large signal-to-noise ratio (SNR) values, the two bounds coincide. This observation enables us to compute the sum-rate multiplexing gain (SMG) of the system. Subsequently, we show how each user can regulate its rate to guarantee fairness while maximizing SMG. We compare the FH and FD systems in terms of the following performance measures: average sum-rate multiplexing gain (η_1), average multiplexing gain per user (η_2), the minimum multiplexing gain per user (η_3), average diversity exponent for each user (η_4) and service capability. We show that (depending on the probability mass function of the number of active users), the FH system can offer a significant improvement in terms of η_1 and η_2 (implying a more efficient usage of the spectrum) and also in terms of η_4 (implying a higher reliability). It is also shown that $\frac{1}{e} \leq \frac{\eta_3^{(FH)}}{\eta_3^{(FD)}} \leq 1$, i.e., the loss incurred in η_3 is not more than $\frac{1}{e}$. Finally, computation of the so-called service capability shows that in FH systems any number of users can coexist fairly, while the maximum number of users in FD system is limited by the number of available bands.

Index Terms

Spectral Efficiency, Decentralized Networks, Mixed Gaussian Interference, Diversity-Multiplexing Tradeoff

I. Introduction

Optimal resource allocation is an imperative issue in wireless networks. When multiple users share the same spectrum, the destructive effect of multi-user interference can limit the achievable rates. As such, an effective and low complexity frequency sharing strategy which maximizes the degrees of freedom per user, while mitigating the impact of the multi-user interference is desirable. In frequency division (FD) systems, different users transmit over disjoint frequency bands. Due to practical considerations, such FD systems usually rely on a fixed number of such frequency bands. The main drawback of FD systems is that most of the time the majority of the potential users may be inactive, reducing the resulting spectral efficiency. Reference [1] considers a network of several users with mutual interference. Treating the interference as noise, a central controller computes the optimum power allocation of each link over the spectrum to maximize a global utility function. This leads to the best spectrum sharing strategy for a specific number of users. Clearly, if the number of users changes, the system is not guaranteed to offer the best possible spectral efficiency. In fact, it is shown in [1] that if the crossover gains are sufficiently greater than the forward gains, the frequency division is optimum. However, as mentioned earlier, if the number of users sharing the spectrum is random, FD systems can be highly inefficient in terms of the overall spectral efficiency. To avoid the need for a central controller, cognitive radios [2] are introduced which can sense the bands and transmit over an unoccupied portion of the available spectrum. Fundamental limits of wireless networks with cognitive radios are studied in [3]-[5]. Although cognitive radios avoid the use of a central controller, they require methods for frequency sending and dynamic frequency assignment which add to the overall system complexity. Regardless of the complexity issue, FD systems (with or without cognitive radios) can have poor spectrum efficiency when the number of active users is significantly below its maximum possible value. Noting the above points, it is desirable to have a decentralized frequency sharing strategy (without the need for cognitive radios) which allows the users to coexist, while utilizing the spectrum efficiently and fairly.

Motivated by the above observations, we consider a decentralized network operating on a set of u frequency sub-bands to be shared among n users. Different transmitters are linked to different receivers through paths with static and non-frequency-selective fading. Each user is assumed to have no prior knowledge about the code-books of the other users. We propose a frequency hopping (FH) strategy in which the i^{th} user selects v_i frequency sub-bands among the u available sub-bands and hops to another set of v_i sub-bands for the next transmission. It is assumed that all users transmit independent Gaussian code-books over their chosen frequency bands.

As each user hops over different subsets of the sub-bands without informing other users about its hopping pattern, sensing the spectrum to track the instantaneous interference is a difficult task. This assumption makes the interference probability density function (PDF) on each frequency sub-band at the receiver side of each user be mixed Gaussian. Since the channel gains have a continuous PDF, the number of Gaussian components in the interference PDF is 2^{n-1} with probability one. Each user is able to derive the interference PDF after a sufficiently long training period. Being a random variable, the number of active users in the system is taken to be a global knowledge as it can be inferred from the number of interference levels.

We derive upper and lower bounds on the achievable rate of each user which coincide in the high SNR regime. This enables us to obtain the sum-rate multiplexing gain of the network. We show how each user can regulate its rate close to the achievable rate within a gap which saturates as SNR increases. In fact, the only information each transmitter needs are the highest interference level at its affiliated receiver and its forward channel gain.

We compare the centralized FD with the FH system based on five measures namely, average sum-rate multiplexing gain (η_1), average multiplexing gain per user (η_2), minimum multiplexing gain per use (η_3), average diversity exponent for each user (η_4) and service capability where the latter is the average of the fraction of users who are getting service out of the total number of active users.

We show cases (depending on the probability mass function of the number of active users) where the FH system offers larger values of η_1 and η_2 implying more efficient frequency usage. In fact, the FD system is already designed to service up to $K \leq u$ users where $K|u$. The central controller divides the sub-bands into K clusters each containing $\frac{u}{K}$ sub-bands. Each cluster is assigned to a user. For example, if there is only one active user in the system, $\frac{K-1}{K}u$ frequency bands are unused. But, the FH scheme allows this active user to spread its power on the whole band achieving a higher spectral efficiency.

On the other hand, since the FD system is designed to handle the case where the number of active users is K, the minimum multiplexing gain per user is $\eta_3^{(FD)} = \frac{u}{K}$. As we will see, $\eta_3^{(FH)} = \frac{u}{K}(1 - \frac{1}{K})^{K-1}$ which is less than $\eta_3^{(FD)}$. But, one can easily show that $\frac{\eta_3^{(FH)}}{\eta_3^{(FD)}} \geq \frac{1}{e}$ for all K, i.e., the loss incurred in the

FH system in terms of η_3 is not more than $\frac{1}{e}$.

In case no transmitter has the necessary knowledge about the channel gains to regulate its rate, we derive lower bounds on the diversity-multiplexing tradeoff [5] of each active user. It is shown that depending on the distribution of the number of active users, FH outperform FD in terms of η_4, i.e., it provides each user with significantly more reliability. Also, it might happen that there are more than u users in the system. Clearly, FD system is not capable to provide service for all of these users. On the other hand, the FH system allows all of these users to share the spectrum. This is interpreted as a higher service capability.

The paper outline is as follows. System model is given in section II. In section III, upper bounds on the achievable rates of users are computed. Section IV offers lower bounds on the achievable rates of users. In section V, based on the results in sections III and IV, we discuss how the users in the FH system fairly share the band while maximizing the multiplexing gain per user. Comparison between the FH and FD systems is given in this section. in sections VI and VII, we discuss about generalizing the FH system and the dual of FH in time respectively. We conclude the paper in section VIII where outage consideration and diversity-multiplexing tradeoff of the FH system is offered. We use the notation $f(\gamma) \sim g(\gamma)$ implying $\lim_{\gamma\to\infty} \frac{f(\gamma)}{g(\gamma)} = 1$ throughout the paper.

II. System Model

We consider a communication system with n users where the i^{th} user exploits $v_i(\leq u)$ out of the u sub-bands and spreads its available power, P, equally over these selected bands by transmitting independent Gaussian signals of variance $\frac{P}{v_i}$ over each of the chosen sub-bands. This user hops to another set of v_i frequency sub-bands after each transmission. We denote the achievable rate of the i^{th} user by R_i. The static and non frequency-selective fading coefficient of the link connecting the i^{th} transmitter to the j^{th} receiver is shown by $h_{i,j}$. Each receiver knows already the hopping pattern of its affiliated transmitter. On the other hand, as all users hop over different portions of the spectrum from transmission to transmission, no user is assumed to be capable of tracking the instantaneous interference. This assumption makes the interference plus noise PDF at the receiver side of each user be a mixed Gaussian distribution. In fact, depending on different choices the other users make to select the frequency sub-bands and values of the crossover gains, this mixed Gaussian distribution has up to 2^{n-1} power levels. For each i, the channel model for the i^{th} link is as follows:

$$\vec{Y}_i = h_{i,i}\vec{X}_i + \vec{Z}_i \tag{1}$$

where $\vec{X}_i$ is the $u \times 1$ input vector of the i^{th} user and $\vec{Z}_i$ is the noise plus interference vector on the receiver side of the i^{th} user. One may write $p_{\vec{X}_i}(\vec{x}) = \sum_{C \in \mathcal{C}} \frac{1}{\binom{u}{v_i}} g(\vec{x}, C)$ where $g(\vec{x}, C)$ denotes a zero-mean jointly Gaussian distribution of covariance matrix C and the set $\mathcal{C}$ includes all $u \times u$ diagonal matrices where v_i out of the u diagonal elements are $\frac{P}{v_i}$ while the rest are zeros. Denoting the noise plus interference on the j^{th} band at the receiver side of the i^{th} user by $Z_{i,j}$ (the j^{th} component of $\vec{Z}_i$), it is clear that $p_{Z_{i,j}}(z)$ is not dependent on j. This is by the fact that crossover gains are not sensitive to frequency and there is no particular interest to a specific frequency sub-band by any user. We assume there are $L_i + 1$ ($L_i \leq 2^{n-1} - 1$) possible non-zero power levels for $Z_{i,j}$, say $\{\sigma_{i,l}^2\}_{l=0}^{L_i}$. The occurrence probability of $\sigma_{i,l}^2$ is denoted by $a_{i,l}$. Then, $p_{Z_{i,j}}(z)$ is a mixed Gaussian distribution as follows:

$$p_{Z_{i,j}}(z) = \sum_{l=0}^{L_i} \frac{a_{i,l}}{\sqrt{2\pi}\sigma_{i,l}} \exp - \frac{z^2}{2\sigma_{i,l}^2} \tag{2}$$

where $\sigma^2 = \sigma_{i,0}^2 < \sigma_{i,1}^2 < \sigma_{i,2}^2 < ... < \sigma_{i,L_i}^2$ (σ^2 is the ambient noise power). In fact, one may write $Z_{i,j} = \sum_{k=1, k \neq i}^{n} \epsilon_{k,j} h_{k,i} X_{k,j} + \nu_{i,j}$ where $X_{k,j}$ is the signal of the k^{th} user sent on the j^{th} sub-band, $\epsilon_{k,j}$ is a Bernoulli random variable showing if the k^{th} user has utilized the j^{th} sub-band and $\nu_{i,j}$ is the ambient noise which is a zero-mean Gaussian random variable with variance σ^2. Obviously, $\Pr\{\epsilon_{k,j} = 1\} = \frac{v_k}{u}$. Also, a quantity of interest would be the following:

$$\begin{aligned} a_{i,0} &= \Pr\{Z_{i,j} \text{ contains no interference}\} \\ &= \prod_{k \neq i} \Pr\{\epsilon_{k,j} = 0\} = \prod_{k \neq i} (1 - \frac{v_i}{u}). \end{aligned} \tag{3}$$

We notice that for each $l \geq 1$, there exists a $c_{i,l} > 0$ such that $\sigma_{i,l}^2 = \sigma^2 + c_{i,l} P$ where $c_{i,1} < c_{i,2} < ... < c_{i,L_i}$. To compute R_i, one may see that for each i, the communication channel of the i^{th} user is a channel with state S_i, the hopping pattern, which is independently changing over different transmissions and is known to both the transmitter and receiver ends of the i^{th} user. The achievable rate of such a channel is given by

$$R_i = I(\vec{X}_i; \vec{Y}_i | S_i) = \sum_{s_i \in \mathcal{S}_i} \Pr(S_i = s_i) I(\vec{X}_i; \vec{Y}_i | S_i = s_i) \tag{4}$$

where $I(\vec{X}_i; \vec{Y}_i | S_i = s_i)$ is the mutual information between $\vec{X}_i$ and $\vec{Y}_i$ for the specific sub-band selection dictated by $S_i = s_i$. The set $\mathcal{S}_i$ denote all possible selections of v_i out of the u sub-bands. As $p_{\vec{Z}_i}(\vec{z})$ is a symmetric density function, meaning all its components have the same PDF given in (2), we deduce that

$I(\vec{X}_i;\vec{Y}_i|S_i=s_i)$ is independent of s_i. Therefore, we may assume any specific sub-band selection for the i^{th} user in $\mathcal{S}_i$, say the first v_i out of the u sub-bands. Denoting this specific state by s_i^*, we get:

$$R_i = I(\vec{X}_i;\vec{Y}_i|S_i=s_i^*) \tag{5}$$

In this case, we denote $\vec{Y}_i$ and $\vec{X}_i$ by $\vec{Y}_i(s_i^*)$ and $\vec{X}_i(s_i^*)$ respectively. Obviously, we have:

$$R_i = I(\vec{X}_i(s_i^*);\vec{Y}_i(s_i^*)) = \mathrm{h}(\vec{Y}_i(s_i^*)) - \mathrm{h}(\vec{Z}_i). \tag{6}$$

Throughout the paper, the number of users is assumed to be a random variable. To decode the data, the receiver of the i^{th} user is expected to know the noise plus interference PDF, $p_{\vec{Z}_i}(\vec{z})$, after a sufficiently long training period. As we will see, each transmitter can regulate its rate close to its achievable rate within a gap which is bounded in terms of SNR. To do this, each transmitter only needs to know the greatest interference level on each frequency sub-band at the receiver side and its forward channel gain. In case $n=2$, we show even without the knowledge about the greatest interference term, it is possible to regulate the rate at the transmitters to achieve the ultimate multiplexing gain per user in the high SNR regime. We conjecture this to be true for all n. Clearly, if the gains $\{h_{i,j}\}$ have a continuous distribution, the number of interference levels is equal to 2^{n-1} with probability one. As such, n is also assumed to be a global knowledge among users.

III. Upper Bounds on The Achievable Rates

In this section, we develop an upper bound, R_i^{ub}, on the achievable rate of the i^{th} user which is tight enough to ensure that $R_i^{ub} - R_i$ does not increase unboundedly as SNR increases. The idea behind this upper bound is the convexity of R_i in terms of $p(\vec{Y}_i(s_i^*)|\vec{X}_i(s_i^*))$.

Let $\vec{W}_i$ be the $u \times 1$ interference vector where its j^{th} component, $W_{i,j}$, is a random variable showing the interference term on the j^{th} frequency band at the receiver. In terms of our previous notation, $W_{i,j} = \sum_{k=1,k\neq i}^{n} \epsilon_{k,j} h_{k,i} X_{k,j}$. Clearly, $\vec{W}_i$ is a mixed Gaussian vector where its Gaussian components represent different choices the other users make to select their sub-bands. In fact, we have $p_{\vec{W}_i}(\vec{w}) = \frac{1}{M_i}\sum_{m=1}^{M_i} g(\vec{w}, D_{i,m})$ where $M_i = \prod_{j\neq i}\binom{u}{v_j}$, and as each user transmits independent Gaussian signals over its chosen sub-bands, the matrices $\{D_{i,m}\}_{m=1}^{M_i}$ are diagonal, i.e., $D_{i,m} = diag(d_{i,m}^{(1)}, \cdots, d_{i,m}^{(u)})$. If the probability density function of the interference vector consisted only of $g(\vec{w}, D_{i,m})$, the forward link of the i^{th} channel would be converted into an additive Gaussian channel. The achievable rate of such a virtual channel is simply given by:

$$R_{i,m} = \frac{1}{2}\log\frac{\det(\mathrm{Cov}(\vec{X}_i(s_i^*)) + D_{i,m} + \sigma^2 I_u)}{\det(D_{i,m} + \sigma^2 I_u)}$$

$$= \frac{1}{2}\log\frac{\prod_{j=1}^{v_i}(\frac{|h_{i,i}|^2 P}{v_i} + d_{i,m}^{(j)} + \sigma^2)}{\prod_{j=1}^{v_i}(d_{i,m}^{(j)} + \sigma^2)} = \frac{1}{2}\sum_{j=1}^{v_i}\log\left(1 + \frac{|h_{i,i}|^2 P}{v_i(d_{i,m}^{(j)} + \sigma^2)}\right). \tag{7}$$

Let us state this more concisely as follows. let $T_{i,m} = \{j | 1 \leq j \leq v_i, d_{i,m}^{(j)} = 0\}$. Defining $\gamma = \frac{P}{\sigma^2}$, we get:

$$R_{i,m} = \frac{|T_{i,m}|}{2}\log\left(1 + \frac{|h_{i,i}|^2\gamma}{v_i}\right) + \tilde{R}_{i,m} \tag{8}$$

where

$$\tilde{R}_{i,m} = \frac{1}{2}\sum_{1\leq j\leq v_i : d_{i,m}^{(j)}\neq 0}\log\left(1 + \frac{|h_{i,i}|^2 P}{v_i(d_{i,m}^{(j)} + \sigma^2)}\right). \tag{9}$$

As each non-zero $d_{i,m}^{(j)}$ is proportional to P, it is clear that $\lim_{\gamma\to\infty}\tilde{R}_{i,m} < \infty$. We know that R_i is convex in terms of $p_{\vec{Y}_i|\vec{X}_i}(\vec{y}|\vec{x}) = p_{\vec{Z}_i}(\vec{y} - \vec{x})$. On the other hand, $p_{\vec{Z}_i}(\vec{z}) = \frac{1}{M_i}\sum_{m=1}^{M_i} g(\vec{z}, D_{i,m} + \sigma^2 I_u)$. Therefore, we have:

$$R_i \leq \frac{1}{M_i}\sum_{m=1}^{M_i} R_{i,m}$$

$$= \left(\frac{1}{M_i}\sum_{m=1}^{M_i}|T_{i,m}|\right)\frac{1}{2}\log\left(1 + \frac{|h_{i,i}|^2\gamma}{v_i}\right) + \tilde{R}_i \tag{10}$$

where $\tilde{R}_i = \frac{1}{M_i}\sum_{m=1}^{M_i}\tilde{R}_{i,m}$. Clearly, as each $\tilde{R}_{i,m}$ saturates by increasing γ, one has $\lim_{\gamma\to\infty}\tilde{R}_i < \infty$. The following lemma offers an explicit expression for $\frac{1}{M_i}\sum_{m=1}^{M_i}|T_{i,m}|$.

Lemma 1

$$\frac{1}{M_i}\sum_{m=1}^{M_i}|T_{i,m}| = v_i\prod_{k=1, k\neq i}^{n}\left(1 - \frac{v_k}{u}\right)$$

Proof: Defining $A_j = \{m : |T_{i,m}| = j\}$, one may express the left side as:

$$\frac{1}{M_i}\sum_{m=1}^{M_i}|T_{i,m}| = \frac{1}{M_i}\sum_{j=1}^{v_i} j|A_j|$$

Let F be a random variable showing the number of interference free bands among the v_i bands selected by the i^{th} user. Noting that $\Pr\{F=j\} = \frac{|A_j|}{M_i}$, we have:

$$\frac{1}{M_i}\sum_{m=1}^{M_i}|T_{i,m}| = \sum_{j=1}^{v_i} j \Pr\{F=j\} = E\{F\}$$

Let us define:

$$F_j = \begin{cases} 1 & W_{i,j}=0 \\ 0 & W_{i,j}\neq 0 \end{cases}$$

obviously, $F = \sum_{j=1}^{v_i} F_j$. As such, we get:

$$E\{F\} = \sum_{j=1}^{v_i} E\{F_j\} = \sum_{j=1}^{v_i} \Pr\{W_{i,j}=0\}$$

but $\forall j : \Pr\{W_{i,j}=0\} = a_{i,0} = \prod_{k=1,k\neq i}^{n}\left(1-\frac{v_k}{u}\right)$ which yields:

$$E\{F\} = v_i \prod_{k=1,k\neq i}^{n}\left(1-\frac{v_k}{u}\right).$$

■

Based on (10) and lemma 1, we propose the following theorem:

Theorem 1 *There exists an upper bound R_i^{ub} on the achievable rate of the i^{th} user given by*

$$R_i^{ub} = \frac{1}{2}v_i \prod_{k=1,k\neq i}^{n}\left(1-\frac{v_k}{u}\right)\log\gamma + \tilde{R}_i$$

where $\lim_{\gamma\to\infty}\tilde{R}_i < \infty$. In particular, $R_i^{ub} \sim \frac{1}{2}v_i \prod_{k=1,k\neq i}^{n}\left(1-\frac{v_k}{u}\right)\log\gamma$

As we will see in the next section, $\lim_{\gamma\to\infty} R_i^{ub} - R_i < \infty$.

IV. Lower Bounds on the Rates

In this section, we proceed to obtain a lower bound, R_i^{lb}, on the achievable rate of the i^{th} user which has interestingly the same asymptotic expression as that of R_i^{ub} obtained in the previous section. This enables us to deduce the asymptotic expression for R_i itself. The idea behind deriving this lower bound is to invoke entropy power inequality (EPI) which was first used by Shannon as a means of getting a lower bound on the capacity. As we will see, this initial lower bound is not in a closed form as it depends on the entropy of a mixed Gaussian random variable. In appendix A, through a careful examination of such an entropy, we obtain an appropriate upper bound on it which leads us to the final lower bound R_i^{lb}.

We define $\vec{X}_i'$ to be the $v_i \times 1$ signal vector of the first transmitter which is sent through the first v_i chosen frequency bands. Let $\vec{Y}_i' = h_{i,i}\vec{X}_i' + \vec{Z}_i'$, where $\vec{Z}_i'$ is the noise plus interference vector at the receiver side on the first v_i frequency bands. According to EPI, we have:

$$2^{\frac{2}{v_i}\mathrm{h}(\vec{Y}_i')} \geq 2^{\frac{2}{v_i}\mathrm{h}(h_{i,i}\vec{X}_i')} + 2^{\frac{2}{v_i}\mathrm{h}(\vec{Z}_i')}. \tag{11}$$

Dividing both sides by $2^{\mathrm{h}(\vec{Z}_i')}$, we get:

$$\mathrm{h}(\vec{Y}_i') - \mathrm{h}(\vec{Z}_i') \geq \frac{v_i}{2}\log(2^{\frac{2}{v_i}\left(\mathrm{h}(h_{i,i}\vec{X}_i')-\mathrm{h}(\vec{Z}_i')\right)} + 1). \tag{12}$$

On the other hand, since $\vec{Y}_i'$ is a subvector of $\vec{Y}_i(s_i^*)$, we have:

$$R_i = I(\vec{X}_i(s_i^*); \vec{Y}_i(s_i^*)) \geq I(\vec{X}_i'; \vec{Y}_i') = \mathrm{h}(\vec{Y}_i') - \mathrm{h}(\vec{Z}_i'). \tag{13}$$

Based on (12) and (13), we get the following lower bound on R_i:

$$R_i \geq \frac{v_i}{2}\log(2^{\frac{2}{v_i}\left(\mathrm{h}(h_{i,i}\vec{X}_i')-\mathrm{h}(\vec{Z}_i')\right)} + 1). \tag{14}$$

As $\vec{Z}_i'$ is a mixed Gaussian vector, there is no closed-form formula for $\mathrm{h}(\vec{Z}_i')$. To circumvent this difficulty, we have to find an appropriate upper bound on $\mathrm{h}(\vec{Z}_i')$. A general upper bound on the entropy of a random vector is the entropy of a Gaussian vector of the same covariance matrix. But, it can be verified that this yields a lower bound on R_i which is less than a constant threshold for all values of γ, and hence would not be suitable for our purposes. To find a sufficiently tight upper bound on $\mathrm{h}(\vec{Z}_i')$, we must investigate the exact PDF of $\vec{Z}_i'$. Using the chain rule for the entropy function, one has the following bound:

$$\mathrm{h}(\vec{Z}_i') \leq \sum_{j=1}^{v_i} \mathrm{h}(Z_{i,j}) \tag{15}$$

It is notable that components of $\vec{Z}_i'$ are not independent. Thus, the above bound is actually strict. The following proposition yields an upper bound on $\mathrm{h}(Z_{i,j})$.

Proposition 1 *For every $1 \leq j \leq v_i$ and for all values of γ, there exists an upper bound on $\mathrm{h}(Z_{i,j})$ given by*

$$\mathrm{h}(Z_{i,j}) \leq \frac{1}{2}(1 - a_{i,0})\log(c_{i,L_i}\gamma + 1) + \log(\sqrt{2\pi e}\sigma) + \kappa_i$$

where $\kappa_i = -a_{i,0}\log a_{i,0} - (1 - a_{i,0})\log a_{i,L_i} + \frac{1}{2}a_{i,0}\log e$ is a term not depending on γ and the channel *gains. Also, we have:*

$$\mathrm{h}(Z_{i,j}) \sim \frac{1}{2}(1-a_{i,0})\log\gamma$$

.

Proof: See Appendix A. ■

In fact, the upper bound given on $\mathrm{h}(Z_{i,j})$ replacing $Z_{i,j}$ by a Gaussian random variable of the same variance is asymptotically equivalent to $\frac{1}{2}\log\gamma$. In contrast, the upper bound given in proposition 1, includes the coefficient $1-a_{i,0}$ which makes it tighter. Based on the above proposition, and by (14) and (15), we get:

$$R_i \geq \frac{v_i}{2}\log\left(2^{\frac{2}{v_i}\left(\frac{1}{2}\log(2\pi e\frac{|h_{i,i}|^2P}{v_i})^{v_i}-v_i(\frac{1}{2}(1-a_{i,0})\log(c_{i,L_i}\gamma+1)+\log(\sqrt{2\pi e}\sigma)+\kappa_i)\right)}+1\right)$$

$$= \frac{v_i}{2}\log\left(\frac{2^{2\kappa_i}|h_{i,i}|^2\gamma}{(c_{i,L_i}\gamma+1)^{1-a_{i,0}}}+1\right). \tag{16}$$

But, $\frac{v_i}{2}\log\left(\frac{2^{2\kappa_i}|h_{i,i}|^2\gamma}{(c_{i,L_i}\gamma+1)^{1-a_{i,0}}}+1\right) \sim \frac{1}{2}v_i a_{i,0}\log\gamma$. Thus, we come up with the following theorem of this section:

Theorem 2 *There exists a lower bound R_i^{lb} on the achievable rate of the i^{th} user which satisfies*

$$R_i^{lb} \sim \frac{1}{2}v_i\prod_{k=1,k\neq i}^{n}\left(1-\frac{v_k}{u}\right)\log\gamma.$$

Now, we note the following remarks stating some points regarding the last two theorems.

- Interestingly, R_i^{lb} has the same asymptotic expression as that of R_i^{ub}. This analogy enables us to deduce the asymptotic expression for R_i. Also, it is now clear that R_i^{ub} is tight in the sense that $\lim_{\gamma\to\infty} R_i^{ub}-R_i < \lim_{\gamma\to\infty} R_i^{ub}-R_i^{lb} < \infty$.
- From now on, we assume that the i^{th} transmitter regulates its rate at R_i^{lb}. It can be seen that the only parameters needed to compute R_i^{lb} are $|h_{i,i}|$, c_{i,L_i} and κ_i. As we know, c_{i,L_i} represents the greatest interference level on each frequency band at the i^{th} user receiver side. This must be passed over to the transmitter side via a feedback link. On the other hand, κ_i completely depends on $\{v_j\}_{j=1}^{n}$ along the terms $a_{i,0}=\prod_{j=1,j\neq i}^{n}(1-\frac{v_j}{u})$ and $a_{i,L_i}=\prod_{j=1,j\neq i}^{n}\frac{v_j}{u}$. For example, if u is an integer multiple of n, and we take $v_j=\frac{u}{n}$ for all j, then $a_{i,0}=(1-\frac{1}{n})^{n-1}$, $a_{i,L_i}=\frac{1}{n^{n-1}}$ and $\kappa_i=\frac{1}{2}\log\frac{n^{(n-1)(1-(1-\frac{1}{n})^{n-1})}\exp(1-\frac{1}{n})^{n-1}}{(1-\frac{1}{n})^{(n-1)(1-\frac{1}{n})^{n-1}}}$. In fact, as we will see in the next section, setting $v_j=\frac{u}{n}$ for all j is towards maximizing the sum-rate multiplexing gain.

• In appendix B, for a system with two users, i.e., $n = 2$, via different bounding techniques from those offered already, R_i^{ub} and R_i^{lb} are computed as follows:

$$R_i^{lb} = \frac{1}{2}(v_i - \frac{v_1 v_2}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v_i}) + \frac{1}{2}\frac{v_1 v_2}{u} \log(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1 + \frac{|h_{i',i}|^2\gamma}{v_{i'}}}) \tag{17}$$

and

$$R_i^{lb} = \frac{1}{2}(v_i - \frac{v_1 v_2}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v_i}) + \frac{1}{2}\frac{v_1 v_2}{u} \log(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1 + \frac{|h_{i',i}|^2\gamma}{v_{i'}}}) - \kappa_i' \tag{18}$$

where $i' = 3 - i$ for $i \in \{1, 2\}$ and $\kappa_i' = -u(a_{i,0} \log a_{i,0} + (1 - a_{i,0}) \log(1 - a_{i,0}))$. It is seen that both bounds are all the same up to a constant difference κ_i' which is not dependent on γ and the channel gains. In particular, we see that $R_i' = (v_i - \frac{v_0 v_1}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v_i}) - \kappa_i'$ is an achievable rate (it is less than R_i^{lb}) as far as γ is large enough to ensure that $R_i' > 0$. This can be computed knowing $|h_{i,i}|$. This achievable rate is asymptotically equivalent to R_i^{ub}. Therefore, we have found a computable rate achieving the full multiplexing gain which is not dependent on c_{i,L_i}. We conjecture that the same result holds for general n.

V. System Design

In this section, we consider the complex case where signals, ambient noise and channel gains are circular complex Gaussian random variables. This affects the previous results via multiplication by a factor of two. In general, there are two fixed parameters in the system, the number of frequency bands, u, and the maximum number of active users that the system is designed to handle, K. We compare the FH system with the centralized FD system according to five key measures to be defined later. Based on the results in the previous sections, there exist upper and lower bounds on the achievable rate of each user which coincide in the high SNR regime. Thus, the achievable rate itself must be asymptotically equivalent to each of these bounds, i.e.,

$$R_i \sim v_i \prod_{k=1, k\neq i}^{n} \left(1 - \frac{v_k}{u}\right) \log \gamma. \tag{19}$$

Let $SR = \sum_{i=1}^{n} R_i$ be the sum-rate. Then,

$$SR \sim r_{SR} \log \gamma \tag{20}$$

where

$$r_{SR} = \sum_{i=1}^{n} v_i \prod_{1 \le k \le n, k \ne i} \left(1 - \frac{v_k}{u}\right). \tag{21}$$

We call r_{SR} the sum-rate multiplexing gain of the system. r_{SR} is a symmetric function of v_i's. In a "fair" FH system, it is required that $v_i = v$ for all i. Thus,

$$r_{SR} = nv\left(1 - \frac{v}{u}\right)^{n-1}. \tag{22}$$

Maximizing this in terms of v yields:

$$v_{opt} = \begin{cases} 1 & \text{if } \frac{u}{n} < 1 \\ \lfloor \frac{u}{n} \rfloor & \text{if } \frac{u}{n} \ge 1 \end{cases}. \tag{23}$$

In the sequel, we compare the performance of the FD system with that of the decentralized network adopting the FH strategy. We assume the number of users in the system is a random variable N with probability mass function $q_n = \Pr\{N = n\}$ for $n \ge 1$. For the moment, we assume $\Pr\{N > K\} = 0$. In what follows all expectations are with respect to the number of active users. Our comparison is based on five performance measures namely, average sum-rate multiplexing gain, average multiplexing gain per user, the minimum multiplexing gain per user, average diversity exponent for each user and the so-called service capability. Service capability shows the fraction of users getting service among all the active users in the system.

- Average sum-rate multiplexing gain

Average sum-rate multiplexing gain is defined as $\eta_1 = E\{r_{SR}\}$. The FD system is already designed to handle $K \le u$ users where $K|u$. The frequency sub-bands are divided into K clusters each containing $\frac{u}{K}$ sub-bands. Each user that enters the system looks for an empty cluster. If there is one, the user occupies the cluster. If there is no empty cluster, no service is available. Therefore, the sum-rate multiplexing gain is

$$r_{SR}^{(FD)} = \begin{cases} N\frac{u}{K} & N \le K \\ u & N > K \end{cases}. \tag{24}$$

On the other hand, in a decentralized network with FH strategy, the parameter K is meaningless. In fact, by the nature of FH, any number of active users can get service. Since N is a global knowledge, by (22) and (23), $r_{SR}^{(FH)}$ is given by:

$$r_{SR}^{(FH)} = \begin{cases} N\lfloor \frac{u}{N} \rfloor (1 - \frac{1}{u}\lfloor \frac{u}{N} \rfloor)^{N-1} & N \le u \\ N(1-\frac{1}{u})^{N-1} & N > u \end{cases} \tag{25}$$

Example 1 *Assume there are always at most three active users in the system and $2|u$. As such, the central controller in the FD system sets $K = 2$, and according to (24), we have $\eta_1^{(FD)} = E\{r_{SR}^{(FD)}\} = q_1\frac{u}{2} + q_2 u$. On the other hand, based on (25), we get $\eta_1^{(FH)} = E\{r_{SR}^{(FH)}\} = q_1 u + 2q_2\frac{u}{2}(1-\frac{1}{u}\frac{u}{2}) = q_1 u + q_2\frac{u}{2}$. Therefore, as far as $q_1 u + q_2\frac{u}{2} > q_1\frac{u}{2} + q_2 u$ or equivalently $q_1 > q_2$, we have $\eta_1^{(FH)} > \eta_1^{(FD)}$. Thus, if $q_1 > \frac{1}{2}$, i.e., the probability that two users become active simultaneously is less than $\frac{1}{2}$, the FH system utilizes the band more efficiently.*

- Average multiplexing gain per user

Average multiplexing gain per user is defined as $\eta_2 = E\{\frac{r_{SR}}{N}\}$. This measure shows the multiplexing gain each user achieves on average.

Example 2 *Considering the same setup as in example 1, we have $\eta_2^{(FD)} = \frac{u}{2}$ and $\eta_2^{(FH)} = q_1 u + q_2\frac{u}{2}(1 - \frac{1}{u}\frac{u}{2}) = q_1 u + q_2\frac{u}{4}$. Therefore, as far as $q_1 u + q_2\frac{u}{4} > \frac{u}{2}$ or equivalently $q_1 > \frac{1}{3}$, we have $\eta_2^{(FH)} > \eta_2^{(FD)}$. This example together with example 1 show that as far as $q_1 > \frac{1}{2}$, the FH system outperforms the FD system in terms of both η_1 and η_2.*

Example 3 *We consider a decentralized system where at most three users might show up. For simplicity, we assume $6|u$. The FD controller sets $K = 3$. Thus, we have $\eta_1^{(FD)} = q_1\frac{u}{3} + q_2\frac{2u}{3} + q_3 u$ and $\eta_1^{(FH)} = q_1 u + q_2 u(1 - \frac{1}{u}\frac{u}{2}) + q_3 u(1 - \frac{1}{u}\frac{u}{3})^2 = q_1 u + q_2\frac{u}{2} + q_3\frac{4u}{9}$. Hence, as far as $26q_1 + 11q_2 > 14$, we get $\eta_1^{(FH)} > \eta_2^{(FD)}$. Also, $\eta_2^{(FD)} = \frac{u}{3}$ and $\eta_2^{(FH)} = q_1 u + q_2\frac{u}{2}(1 - \frac{1}{u}\frac{u}{2}) + q_3\frac{u}{3}(1 - \frac{1}{u}\frac{u}{3})^2 = q_1 u + q_2\frac{u}{4} + q_3\frac{4u}{27}$. Therefore, if $104q_1 + 23q_2 > 32$, we have $\eta_2^{(FH)} > \eta_2^{(FD)}$.*

- Minimum multiplexing gain per user

The minimum multiplexing gain per user is the smallest possible multiplexing gain that a user attains. We denote this by η_3. Clearly, this happens when there are exactly K active users in the system. As the FD system is already designed to handle the case where K users are present in the system, the minimum multiplexing gain per user is automatically higher. Setting $N = K$, we have $\eta_3^{(FD)} = \frac{r_{SR}^{(FD)}}{K} = \frac{u}{K}$ and $\eta_3^{(FH)} = \frac{r_{SR}^{(FH)}}{K} = \frac{u}{K}(1 - \frac{1}{K})^{K-1}$ by (24) and (25) respectively. Clearly, $\frac{1}{e} \le \frac{\eta_3^{(FH)}}{\eta_3^{(FD)}} \le 1$ as $(1 - \frac{1}{K})^{K-1}$ approaches $\frac{1}{e}$ from above by increasing K. Therefore, the loss incurred in the FH system is at most $\frac{1}{e}$.

• Average diversity exponent per user

The details about the computations in this part are brought in section VI. Assuming the i^{th} transmitter has not the necessary knowledge about channel fading gains to regulate its rate, one may talk about the outage probability of the i^{th} user in the high SNR regime, i.e., $\Pr\{R_i < r\log\gamma\}$ where r is the multiplexing gain of the user. As in [5], we define the diversity exponent for each user as $d = \lim_{\gamma\to\infty}\frac{-\log\Pr\{R_i<r\log\gamma\}}{\log\gamma}$ (d is the same for all i). Then, the average diversity exponent for each user is defined by $\eta_4 = E\{d\}$. A lower bound on $d^{(FH)}$ is obtained in section VI as follows:

$$d^{(FH)} \geq \begin{cases} \left((1-\frac{v_{opt}}{u})^{n-1} - \frac{r}{v_{opt}}\right)^{+} & n > 2 \\ \left(1 - \frac{r}{v_{opt}(1-\frac{v_{opt}}{u})^{n-1}}\right)^{+} & n = 1, 2 \end{cases} \tag{26}$$

where v_{opt} is defined in (23) and $(t)^{+} = \max\{0, t\}$ for any t. Based on results in [5], we have $d^{(FD)} = (1 - \frac{r}{\frac{u}{K}})^{+}$.

Example 4 *Adopting the setup in the previous examples, one can easily see that* $\eta_4^{(FH)} \geq q_1(1-\frac{r}{u})^{+} + q_2(1-\frac{r}{\frac{u}{2}(1-\frac{1}{u}\frac{u}{2})})^{+}$. *Assuming* $r \in [0, \frac{u}{4}]$, *we get* $\eta_4^{(FH)} \geq 1 - \frac{(q_1+4q_2)r}{u}$. *On the other hand, we have* $\eta_4^{(FD)} = (1-\frac{2r}{u})^{+}$. *Thus, for* $r \in [0, \frac{u}{4}]$, *as far as* $1 - \frac{(q_1+4q_2)r}{u} > 1 - \frac{2r}{u}$ *or equivalently* $q_1 > \frac{2}{3}$, *we have* $\eta_4^{(FH)} > \eta_4^{(FD)}$. *This together with examples 1 and 2 imply that as far as* $q_1 > \frac{2}{3}$, *FH outperforms FD in terms of* η_1, η_2 *and* η_4.

• Service capability

Service capability demonstrates the fraction of users getting service out of the whole present users in the system. Let us denote the number of users getting service by N_s. Therefore, the service capability is computed as $E\{\frac{N_s}{N}\}$. In the FH system, the service capability is always one. But, in the FD system, if $N > u$ then certainly a fraction of users can not share the band. This actually occurs whenever $\Pr\{N > u\} > 0$. In case $\Pr\{N \leq u\} = 1$, both systems have service capability equal to one.

VI. On the Generalization of FH

One may consider a generalization of the FH system called GFH. Let us assume that the users are not restricted to choose a fixed number of frequency sub-bands. In fact, for each transmission, the number of bands can be any number between zero and u, and the probability of choosing $0 \leq v \leq u$ sub-bands is denoted by μ_v. Therefore, each user has two random generators. Te first random generator selects a number $0 \le v \le u$ according to the probability mass function $\{\mu_v\}_{v=0}^{u}$ while the other generator selects v sub-bands among the whole available u bands. This repeats from transmission to transmission. Based on the arguments we made in section II, we have:

$$R_i = \sum_{v=0}^{u} \mu_v I(\vec{X}_i(s_{i,v}^*); \vec{Y}_i(s_{i,v}^*)) \tag{27}$$

where $s_{i,v}^*$ denotes the state where the i^{th} user selects the first v sub-bands. Clearly, $I(\vec{X}_i(s_{i,0}^*); \vec{Y}_i(s_{i,0}^*)) = 0$. On the other hand, for any $1 \le i \le n$, we have:

$$\begin{aligned} a_{i,0} &= \Pr\{Z_{i,j} \text{ contains no interference}\} \\ &= \sum_{v_1=0}^{u} \sum_{v_2=0}^{u} \cdots \sum_{v_{n-1}=0}^{u} \prod_{k=1}^{n-1} \mu_{v_k}(1 - \frac{v_k}{u}) = \prod_{k=1}^{n-1} \sum_{v_k=0}^{u} \mu_{v_k}(1 - \frac{v_k}{u}) \\ &= \left(\sum_{v=0}^{u} \mu_v (1 - \frac{v}{u}) \right)^{n-1} = (1 - \frac{\bar{v}}{u})^{n-1} \end{aligned} \tag{28}$$

where $\bar{v} = \sum_{v=0}^{u} \mu_v v$. Based on the results of theorems 1 and 2, we have:

$$I(\vec{X}_i(s_{i,v}^*); \vec{Y}_i(s_{i,v}^*)) \sim v a_{i,0} \log \gamma. \tag{29}$$

Hence, (27) and (29) yield the sum-rate multiplexing gain of GFH as follows:

$$r_{SR}^{(GFH)} = n \sum_{v=0}^{u} \mu_v v (1 - \frac{\bar{v}}{u})^{n-1} = n\bar{v}(1 - \frac{\bar{v}}{u})^{n-1}. \tag{30}$$

This demonstrates that the maximum sum-rate multiplexing gain is actually achieved by the simple FH scheme.

VII. Time-hopping vs Time-division

One may adopt the same hopping idea in time as a dual to frequency hopping. We assume all users transmit on one frequency band. Each user has an on-off transmission pattern which is modeled as an i.i.d. Bernoulli process of parameter β. Therefore, the channel model for the i^{th} user is the following:

$$Y_i = \xi_i h_{i,i} X_i + Z_i \tag{31}$$

where ξ_i is a Bernoulli random variable of parameter β and $Z_i = \sum_{k \neq i} \xi_k h_{k,i} X_k + \eta_i$. Assuming Gaussian signals, the achievable rate of the i^{th} user is given by

$$R_i = I(X_i; Y_i | \xi_i) = \beta I(X_i; h_{i,i} X_i + Z_i). \tag{32}$$

Based on the results of theorems 1 and 2, we have:

$$I(X_i; h_{i,i} X_i + Z_i) \sim (1 - \beta)^{n-1} \log \gamma. \tag{33}$$

Therefore, the sum-rate multiplexing gain is given by:

$$r_{SR} = n\beta(1 - \beta)^{n-1}. \tag{34}$$

Maximizing this in terms of β yields $\beta_{opt} = \frac{1}{n}$. Thus, the sum-rate multiplexing gain of the time hopping (TH) system is given by $r_{SR}^{(TH)} = (1 - \frac{1}{n})^{n-1}$ for $n > 1$ and $r_{SR}^{(TH)} = 1$ for $n = 1$. Comparing this to the time-division (TD) system which is designed to service K users, one can easily find probability mass functions on the number of actives users where the TH system outperforms TD in terms of average sum-rate multiplexing gain.

VIII. Diversity-Multiplexing Tradeoff and Outage Consideration

As we mentioned in section IV, each user requires to know the gain of its forward channel and the greatest interference term on its affiliated receiver to regulate its rate within the achievable region. Throughout this section, we assume there is no channel state information regarding these quantities at the transmitter sides. A common method to assess the performance of the network in this case is to evaluate the outage probability. We assume all crossover gains are modeled as i.i.d circularly complex gaussian random variables. We consider the FH scheme where the input signals are taken to be of circularly complex gaussian distribution. The outage probability for the i^{th} link is given by $\Pr\{R_i < r_i \log \gamma\}$ where r_i is the multiplexing gain of the i^{th} link which by (19) belongs to $[0, v_i a_{i,0}]$. As it was pointed out, there is no closed form for R_i. As such, based on the lower bound derived on R_i in (16), we have:

$$\Pr\{R_i < r_i \log \gamma\} \leq \Pr\{\frac{v_i}{2} \log \left(\frac{2^{2\kappa_i} |h_{i,i}|^2 \gamma}{(c_{i,L_i} \gamma + 1)^{1 - a_{i,0}}} + 1 \right) < r_i \log \gamma\} \tag{35}$$

In fact, $c_{i,L_i} = \sum_{k=1, k \neq i}^{n} |h_{k,i}|^2$ whose distribution is given as $\chi^2_{2(n-1)}$. Let us define the random variables $\alpha_{k,i}$ and $\alpha_i^{(c)}$ by $|h_{k,i}|^2 = \gamma^{-\alpha_{k,i}}$ and $c_{i,L_i} = \gamma^{-\alpha_i^{(c)}}$ respectively. The PDF of these random variables are given in the following lemma:

Lemma 2 *In the high SNR regime,* $p_{\alpha_{i,i}}(\alpha)$ *and* $p_{\alpha_i^{(c)}}(\alpha)$ *are given as*

$$p_{\alpha_{i,i}}(\alpha) = \begin{cases} (\ln \gamma)\gamma^{-\alpha} & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}$$

and

$$p_{\alpha_i^{(c)}}(\alpha) = \begin{cases} (n-1)(\ln \gamma)\gamma^{-(n-1)\alpha} & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}.$$

Proof:

As $h_{k,i}$ is a complex circular gaussian random variable of variance $\frac{1}{2}$ per each dimension, the probability density function of $\alpha_{k,i}$ for each k and i is given by:

$$p(\alpha) = (\ln \gamma)\gamma^{-\alpha} \exp(-\gamma^{-\alpha}). \tag{36}$$

In the high SNR regime as γ goes to infinity, $p(\alpha)$ approaches the following density:

$$p_\infty(\alpha) = \begin{cases} (\ln \gamma)\gamma^{-\alpha} & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}. \tag{37}$$

Let $c_{i,L_i} = \gamma^{-\alpha_i^{(c)}}$. As $c_{i,L_i} = \sum_{k=1,k\neq i}^{n} |h_{k,i}|^2 = \sum_{k=1,k\neq i}^{n} \gamma^{-\alpha_{k,i}}$, we can express $\alpha_i^{(c)}$ in the high SNR regime as follows:

$$\alpha_i^{(c)} = -\max_{k\neq i} -\alpha_{k,i} = \min_{k\neq i} \alpha_{k,i}. \tag{38}$$

As $\alpha_{k,i}$ are i.i.d. with common PDF $p(\alpha)$, we get:

$$p_{\alpha_i^{(c)}}(\alpha) = (n-1)(1 - F_\infty(\alpha))^{n-2} p_\infty(\alpha) \tag{39}$$

where $F_\infty(\alpha)$ is the commulative distribution function (CDF) corresponding to the PDF $p_\infty(\alpha)$, and is given by:

$$F_\infty(\alpha) = \int_{-\infty}^{\alpha} p_\infty(\alpha) d\alpha = \begin{cases} 1 - \gamma^{-\alpha} & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}. \tag{40}$$

This together with (37) and (39) yields the following:

$$p_{\alpha_i^{(c)}}(\alpha) = \begin{cases} (n-1)(\ln\gamma)\gamma^{-(n-1)\alpha} & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}. \tag{41}$$

■

We use the notation $b \doteq \gamma^a$ as an alternative for $\lim_{\gamma\to\infty} \frac{\log b}{\log \gamma} = a$. Therefore, we may write $c_{i,L_i}\gamma + 1 \doteq \gamma^{(1-\alpha_i^{(c)})^+}$ where $(x)^+ = \max\{0, x\}$. By the same token, we get $\frac{|h_{i,i}|^2\gamma}{(c_{i,L_i}\gamma+1)^{(1-a_{i,0})}} + 1 \doteq \gamma^{(1-\alpha_{i,i}-\bar{a}_{i,0}(1-\alpha_i^{(c)})^+)^+}$ where $\bar{a}_{i,0} = 1 - a_{i,0}$. Using this in (35) yields:

$$\Pr\{R_i < r_i \log\gamma\} \leq \Pr\{v_i(1-\alpha_{i,i}-\bar{a}_{i,0}(1-\alpha_i^{(c)})^+)^+ < r_i\}. \tag{42}$$

The following lemma yields the above probability.

Lemma 3 *For* $0 \leq r_i \leq v_i a_{i,0}$,

$$\Pr\{v_i(1-\alpha_{i,i}-\bar{a}_{i,0}(1-\alpha_i^{(c)})^+)^+ < r_i\} \doteq \gamma^{-(a_{i,0}-\frac{r_i}{v_i})}.$$

Proof: Let us define:

$$\mathcal{O} = \{(\alpha_{i,i}, \alpha_i^{(c)}) : v_i(1-\alpha_{i,i}-\bar{a}_{i,0}(1-\alpha_i^{(c)})^+)^+ < r_i\} \tag{43}$$

One can easily see that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$ where

$$\mathcal{O}_1 = \{(\alpha_{i,i}, \alpha_i^c) : \alpha_i^{(c)} > 1, \alpha_{i,i} > 1\},$$

$$\mathcal{O}_2 = \{(\alpha_{i,i}, \alpha_i^c) : \alpha_i^{(c)} > 1, 0 < \alpha_{i,i} < 1, v_i(1-\alpha_{i,i}) < r_i\},$$

$$\mathcal{O}_3 = \{(\alpha_{i,i}, \alpha_i^{(c)}) : 0 < \alpha_i^{(c)} < 1, 1-\alpha_{i,i}-\bar{a}_{i,0}(1-\alpha_i^{(c)}) < 0\}$$

and

$$\mathcal{O}_4 = \{(\alpha_{i,i}, \alpha_i^{(c)}) : 0 < \alpha_i^{(c)} < 1, 0 < v_i(1-\alpha_{i,i}-\bar{a}_{i,0}(1-\alpha_i^{(c)})) < r_i\}.$$

On the other hand, $\mathcal{O}_l \cap \mathcal{O}_{l'} = \varnothing$ for $l \neq l'$. Therefore, we get:

$$\Pr(\mathcal{O}) = \sum_{l=1}^{l=4} \Pr(\mathcal{O}_l). \tag{44}$$

To compute $\Pr(\mathcal{O}_l)$, we proceed as follows. We have:

$$\Pr(\mathcal{O}_l) = \int_{(t_1,t_2)\in\mathcal{O}_l} p_{\alpha_{i,i}}(t_1)p_{\alpha_i^{(c)}}(t_2)dt_1dt_2$$

$$= (n-1)(\ln\gamma)^2 \int_{(t_1,t_2)\in\mathcal{O}_l} \gamma^{-(t_1+(n-1)t_2)} \doteq \gamma^{-d_l} \tag{45}$$

where

$$d_l = \inf_{(t_1,t_2)\in\mathcal{O}_l} t_1 + (n-1)v_2. \tag{46}$$

The last identity follows from Laplace's formula in large deviation theory. One may easily find d_l geometrically by sketching $\mathcal{O}_l$ and the line $t_1 + (n-1)t_2 = t$ for arbitrary t. Increasing t, we look for the first time that the line intersects $\mathcal{O}_l$. The value of t for which this happens is d_l. Following this method, we obtain:

$$d_l = \begin{cases} n & l=1 \\ n - \frac{r_i}{v_i} & l=2 \\ a_{i,0} & l=3 \\ a_{i,0} - \frac{r_i}{v_i} & l=4 \end{cases}. \tag{47}$$

By (44) and (45), we have:

$$\Pr(\mathcal{O}) \doteq \gamma^{-\min_l d_l}. \tag{48}$$

But, (47) gives $\min_l d_l = a_{i,0} - \frac{r_i}{v_i}$. This concludes the lemma.

■

If we define d_i as $\Pr\{R_i < r_i \log\gamma\} \doteq \gamma^{-d_i}$, then by (42) and the above lemma, we get:

$$d_i \geq a_{i,0} - \frac{r_i}{v_i} \tag{49}$$

for $r_i \in [0, a_{i,0}v_i]$. We have shown in appendix B that this lower bound is not in general tight. In fact, appendix B yields the diversity-multiplexing tradeoff for $n=2$ as $d_i = 1 - \frac{r_i}{a_{i,0}v_i}$. Clearly, $a_{i,0} - \frac{r_i}{v_i} < 1 - \frac{r_i}{a_{i,0}v_i}$. To summarize, in a "fair" FH system where $v_1 = ... = v_n = v_{opt}$, we come up with the following for all i:

$$d_i \geq \begin{cases} \left((1-\frac{v_{opt}}{u})^{n-1} - \frac{r}{v_{opt}}\right)^+ & n>2 \\ \left(1 - \frac{r}{v_{opt}(1-\frac{v_{opt}}{u})^{n-1}}\right)^+ & n=1,2 \end{cases}. \tag{50}$$

It would be interesting to compare the FH setup with another scenario in which all users fix their utilized bands and stay unchanged throughout the transmission process, i.e., no random hopping is considered here. We call this scenario fixed band strategy. In a two-user system, each of the two users select x bands out of the u available bands. As the users remain on the selected bands, the receivers of each link might be able two recognize the bands used by the other link. As such, it is reasonable to model the noise plus interference as a gaussian vector in this situation. To compute the outage probability of the first link, we suppose that it has occupied the first v frequency bands. The second link may overlap with the first link over v' of the first v bands. In this case, we denote the achievable rate of the first link by $R_1(v')$. One has the following:

$$\Pr(R_1 < R) = \sum_{v'=0}^{v} \Pr(R_1 < R|v')P_{v'} \tag{51}$$

where R_1 is the achievable rate of the first link, and $p_{v'} = \frac{\binom{v}{v'}\binom{u-v}{v-v'}}{\binom{u}{v}}$ is the probability that v' of the bands selected by the first link are shared with the second link. Clearly, $\Pr(R_1 < R|v') = \Pr(R_1(v') < R)$. We notice that $R_1(v) = u\log\frac{\frac{2P}{v}+\sigma^2}{\frac{P}{v}+\sigma^2}$ which tends to the constant level u as SNR increases. Setting $R = r\log\gamma$, we see that $\Pr(R_1(v) < r\log\gamma) = 1$ in the high SNR regime. Therefore, there appears a constant term $\frac{1}{\binom{u}{v}}$ in the outage probability, and as a consequence, the outage probability does not tend to zero as SNR increases. For the n link setup, the performance is certainly worse. Thus, the frequency hopping scheme surpasses fixed band strategy in terms of outage performance.

IX. Appendix A

In this appendix, we prove propositions 1. Let us consider a general mixed gaussian distribution $p_Z(z)$ with different power levels $\{\sigma_l^2\}_{l=0}^{L}$ and associated probabilities $\{a_l\}_{l=1}^{L}$ given by:

$$p_Z(z) = \sum_{l=0}^{L} \frac{a_l}{\sqrt{2\pi}\sigma_l} \exp -\frac{z^2}{2\sigma_l^2}$$

where $\sigma_l^2 = \sigma^2 + c_l P$ and $0 = c_0 < c_1 < \cdots < c_L$. One may write $p_Z(z)$ as follows:

$$p_Z(z) = \frac{a_L}{\sqrt{2\pi}\sigma_L} \exp(-\frac{z^2}{2\sigma_L^2})\big(1 + \sum_{l=0}^{L-1} \epsilon_l \exp -(\zeta_l z^2)\big) \tag{52}$$

where $\epsilon_l = \frac{a_l\sigma_L}{a_L\sigma_l}$ and $\zeta_l = \frac{1}{2}\left(\frac{1}{\sigma_l^2} - \frac{1}{\sigma_L^2}\right)$. As $\exp -(\zeta_l z^2) \geq \exp -(\zeta_0 z^2)$, taking $b = \sum_{l=0}^{l=L-1} \epsilon_l$, we have:

$$p_Z(z) \geq \frac{a_L}{\sqrt{2\pi}\sigma_L} \exp(-\frac{z^2}{2\sigma_L^2})\big(1 + b\exp -(\zeta_0 z^2)\big). \tag{53}$$

Hence, we get:

$$
\begin{aligned}
I &:= \int p_Z(z) \ln p_Z(z) dz \\
&\geq \left(\ln \frac{a_L}{\sqrt{2\pi}\sigma_L}\right) \int p_Z(z)dz - \frac{1}{2\sigma_L^2} \int z^2 p_Z(z)dz + \int p_Z(z) \ln\left(1 + b\exp -(\zeta_0 z^2)\right)dz \\
&= \ln \frac{a_L}{\sqrt{2\pi}\sigma_L} - \frac{\sum_{l=0}^{L} a_l \sigma_l^2}{2\sigma_L^2} + \sum_{l=0}^{L} J_l
\end{aligned}
\tag{54}
$$

where $J_l = \frac{a_l}{\sqrt{2\pi}\sigma_l} \int \exp -\left(\frac{z^2}{2\sigma_l^2}\right) \ln(1 + b\exp -(\zeta_0 z^2))dz$ for $0 \leq l \leq L$. As each J_l is positive, we get the following lower bound: [1]

$$
I \geq \ln \frac{a_L}{\sqrt{2\pi}\sigma_L} - \frac{\sum_{l=0}^{L} a_l \sigma_l^2}{2\sigma_L^2} + J_0
\tag{55}
$$

To find a proper lower bound on the J_0 in this expression, we proceed as follows:

$$
\begin{aligned}
J_0 &= \frac{a_0}{\sqrt{2\pi}\sigma} \int \exp -\left(\frac{z^2}{2\sigma^2}\right) \ln(1 + b\exp -(\zeta_0 z^2))dz \\
&\geq \frac{2a_0}{\sqrt{2\pi}\sigma} \int_0^\infty \exp -\left(\frac{z^2}{2\sigma^2}\right) \ln(b\exp -(\zeta_0 z^2))dz \\
&= \frac{2a_0 \ln b}{\sqrt{2\pi}\sigma} \int_0^\infty \exp -\frac{z^2}{2\sigma^2} - \frac{2a_0\zeta_0}{\sqrt{2\pi}\sigma} \int_0^\infty z^2 \exp -\left(\frac{z^2}{2\sigma^2}\right) dz \\
&= a_0 \ln b - a_0 \zeta_0 \sigma^2.
\end{aligned}
\tag{56}
$$

In retrospect, we have obtained the following lower bound:

$$
\begin{aligned}
I &\geq \ln\left(\frac{a_L}{\sqrt{2\pi}\sigma_L}\right) - \frac{\sum_{l=0}^{L} a_l \sigma_l^2}{2\sigma_L^2} \\
&\quad + a_0 \ln b - a_0 \zeta_0 \sigma^2 \\
&= \ln \frac{a_L}{\sqrt{2\pi}} - \frac{1}{2}\ln(c_L P + \sigma^2) - \frac{P\sum_{l=0}^{L} c_l a_l + \sigma^2}{2(c_L P + \sigma^2)} \\
&\quad + a_0 \ln\left(\frac{a_0}{a_L}\sqrt{\frac{c_L P + \sigma^2}{\sigma^2}} + \sum_{l=1}^{L-1} \frac{a_l}{a_L}\sqrt{\frac{c_L P + \sigma^2}{c_l P + \sigma^2}}\right) - \frac{1}{2}a_0 - \frac{a_0 \sigma^2}{2\sigma_L^2}.
\end{aligned}
\tag{57}
$$

Thus, the right hand side is asymptotically equivalent to $-\frac{1}{2}\ln P + a_0 \ln \sqrt{P} = -\frac{1}{2}(1 - a_0)\log P$. Since $I = -\frac{h(Z)}{\log e}$, there exists an upper bound on $h(Z)$, namely $f(\gamma)$, which is asymptotically equivalent to

[1] One may show that J_l goes to zero as P goes to infinity for $1 \leq l \leq L$.

$\frac{1}{2}(1-a_0)\log\gamma$. One may simplify the lower bound in (57) as follows. Since $\sum_{l=0}^{L} a_l c_l \le c_L \sum_{l=0}^{L} a_l = c_L$, the term $-\frac{P\sum_{l=0}^{L} c_l a_l + \sigma^2}{c_L P + \sigma^2}$ can be lower bounded by -1. Also, we omit the term $\frac{a_0\sigma^2}{2\sigma_L^2}$ in $-a_0(\frac{1}{\sigma^2} - \frac{1}{\sigma_L^2})\sigma^2$ and the term $\sum_{l=1}^{L-1} \frac{a_l}{a_L}\sqrt{\frac{c_L P+\sigma^2}{c_l P+\sigma^2}}$ inside the $\ln$ function. Thus, we come up with the following:

$$\begin{aligned} \mathrm{h}(Z) &\le \frac{1}{2}\log(c_L\gamma+1) - \frac{1}{2}a_0\log(c_L\gamma+1) - \log\frac{a_L}{\sqrt{2\pi}} + \frac{1}{2}\log\sigma^2 - a_0\log\frac{a_0}{a_L} + \frac{1}{2}(1+a_0)\log e. \\ &= \frac{1}{2}(1-a_0)\log(c_L\gamma+1) + \log(\sqrt{2\pi e}\sigma) + \kappa \end{aligned} \tag{58}$$

where $\kappa = -a_0\log a_0 - (1-a_0)\log a_L + \frac{1}{2}a_0\log e$ is a constant not depending on P. The coefficient c_N is seen to be the only term which might be important, specially in the high SNR regime. To show the second part of proposition 1, it suffices to derive a lower bound on $\mathrm{h}(Z)$ which is asymptotically equivalent to $\frac{1}{2}(1-a_0)\log\gamma$. We know that entropy is a concave functional of the probability density function. Let $g_l(z) = \frac{1}{\sqrt{2\pi}\sigma_l}\exp -\frac{z^2}{2\sigma_l^2}$. As $p_Z(z) = \sum_{l=0}^{L} a_l g_l(z)$, we get:

$$h(Z) \ge \frac{1}{2}\sum_{l=0}^{L} a_l \log(2\pi e\sigma_l^2)$$

$$= \frac{1}{2}\sum_{l=0}^{L} a_l \log(2\pi e(c_l P + \sigma^2)) \sim \frac{1}{2}\Big(\sum_{l=1}^{L} a_l\Big)\log P = \frac{1}{2}(1-a_0)\log P.$$

Hence, there exist a lower bound on $\mathrm{h}(Z)$ which is asymptotically equivalent to $\frac{1}{2}(1-a_0)\log P$.

X. Appendix B

In this appendix, we derive the diversity multiplexing gain tradeoff for $n=2$. As a consequence of this result, we demonstrate that the lower bound on the diversity-multiplexing tradeoff for arbitrary n given in (49) is not tight. We obtain upper and lower bounds on R_i to deduce our result. From now on, we suppose $n=2$.

To get an upper bound on R_i, we use the fact that in general for two random variables U and V, $I(U;V)$ is a convex function of $p_{V|U}(.|.)$ for fixed $p_U(.)$. We investigate two users using the RFH strategy where the i^{th} link exploits v_i bands out of the u bands on each transmission. We suppose $i \in \{0,1\}$ here for notational simplicity. Let us consider a situation where the two links overlap over v^* common frequency bands. In this case, we denote R_i by $R_i(v^*)$ which is given by:

$$R_i(v^*) = (v_i - v^*)\log(1 + \frac{|h_{i,i}|^2\gamma}{v_i}) + v^*\log(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1 + \frac{|h_{i',i}|^2\gamma}{v_{i'}}}) \tag{59}$$

where for a given $i \in \{0,1\}$, we define i' to be $i' = 1-i$. The probability of overlapping of the two users on v^* bands is $p_{v^*} = \frac{\binom{v_i}{v^*}\binom{u-v_i}{v_{i'}-v^*}}{\binom{u}{v_{i'}}}$. Clearly, $0 \le v^* \le \min_k v_k$ and $v_{i'} - v^* \le u - v_i$. As such, we get the following upper bound on R_i:

$$R_i \le \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} p_{v^*} R_1(v^*)$$

$$= \Big(\sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} (v_i - v^*) p_{v^*} \Big) \log(1 + \frac{|h_{i,i}|^2\gamma}{v_i}) + \Big(\sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} v^* p_{v^*} \Big) \log(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1 + \frac{|h_{i',i}|^2\gamma}{v_{i'}}}). \tag{60}$$

As p_{v^*} for $(v_0+v_1-u)^+ \le v^* \le \min_k v_k$ is the hypergeometric probability function, we get

$$\sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} p_{v^*} = 1$$

and

$$\sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} v^* p_{v^*} = \frac{v_0 v_1}{u}.$$

Therefore, we come up with the following upper bound:

$$R_i \le (v_i - \frac{v_0 v_1}{u}) \log(1 + \frac{|h_{i,i}|^2\gamma}{v_i}) + \frac{v_0 v_1}{u} \log(1 + \frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1 + \frac{|h_{i',i}|^2\gamma}{v_{i'}}}). \tag{61}$$

To get a lower bound on R_i, we notice that $R_i = \mathrm{h}(\vec{Y}_i(s_i^*)) - \mathrm{h}(\vec{Z}_i)$. Thus, by finding appropriate lower and upper bounds on $\mathrm{h}(\vec{Y}_i(s_i^*))$ and $\mathrm{h}(\vec{Z}_i)$ respectively, we get a lower bound on R_i.

- Lower Bound on $\mathrm{h}(\vec{Y}_i(s_i^*))$

$p_{\vec{Y}_i(s_i^*)}(\vec{y})$ is given by:

$$p_{\vec{Y}_i(s_i^*)}(\vec{y}) = p_{\vec{X}_i(s_i^*)}(\vec{y}) * p_{\vec{Z}_i}(\vec{y})$$

$$= g(\vec{y}, \mathrm{Cov}(\vec{X}_i(s_i^*))) * \Big(\frac{1}{\binom{u}{v_{i'}}} \sum_{l=1}^{\binom{u}{v_{i'}}} g(\vec{y}, C_l) \Big) = \frac{1}{\binom{u}{v_{i'}}} \sum_{l=1}^{\binom{u}{v_{i'}}} g(\vec{v}, \mathrm{Cov}(\vec{X}_i(s_i^*)) + C_l) \tag{62}$$

where each C_l is a $u \times u$ matrix which has a $v_{i'} \times v_{i'}$ principal sub-matrix equal to $I_{v_{i'}}$ and the rest of its elements are zero. as we know, $\mathrm{Cov}(\vec{X}_i(s_i^*)) = \begin{pmatrix} I_{v_i} & \mathbf{0}_{v_i \times (u-v_i)} \\ \mathbf{0}_{(u-v_i)\times v_i} & \mathbf{0}_{(u-v_i)\times(u-v_i)} \end{pmatrix}$. Let us define the set $\mathcal{B}_{v^*}$ as follows:

$$\mathcal{B}_{v^*} = \{l : \text{There are exactly } v^* \text{ of the first } v_i \text{ diagonal elements of } C_l \text{ equal to one}\}. \tag{63}$$

Thus, we have:

$$p_{\vec{Y}_i(s_i^*)}(\vec{y}) = \frac{1}{\binom{u}{v_{i'}}} \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} \sum_{l \in \mathcal{B}_{v^*}} g(\vec{y}, \mathrm{Cov}(\vec{X}_i(s_i^*)) + C_l). \tag{64}$$

Now, we use the fact that differential entropy is a concave function of the PDF, we obtain the following lower bound:

$$\mathrm{h}(\vec{Y}_i(s_i^*)) \geq \frac{1}{\binom{u}{v_{i'}}} \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} \sum_{l \in \mathcal{B}_{v^*}} \log((2\pi e)^u \det(\mathrm{Cov}(\vec{X}_i(s_i^*)) + C_l)). \tag{65}$$

On he other hand,

$$\begin{aligned} \det(\mathrm{Cov}(\vec{X}_i(s_i^*))+C_l) &= (\frac{|h_{i,i}|^2 P}{v_i}+\sigma^2)^{v_i-v^*}(\frac{|h_{i,i}|^2 P}{v_i}+\frac{|h_{i',i}|^2 P}{v_{i'}}+\sigma^2)^{v^*}(\frac{|h_{i',i}|^2 P}{v_{i'}}+\sigma^2)^{v_{i'}-v^*}(\sigma^2)^{u-v_0-v_1+v^*} \\ &= (\frac{|h_{i,i}|^2 \gamma}{v_i}+1)^{v_i-v^*}(\frac{|h_{i,i}|^2 \gamma}{v_i}+\frac{|h_{i',i}|^2 \gamma}{v_{i'}}+1)^{v^*}(\frac{|h_{i',i}|^2 \gamma}{v_{i'}}+1)^{v_{i'}-v^*}(\sigma^2)^u \end{aligned} \tag{66}$$

for $l \in \mathcal{B}_{v^*}$. Based on (65) and (66), we have:

$$\begin{aligned} \mathrm{h}(\vec{Y}_i(s_i^*)) &\geq u \log(2\pi e \sigma^2) \\ &+ \frac{1}{\binom{u}{v_{i'}}} \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} (v_i - v^*)|\mathcal{B}_{v^*}| \log(\frac{|h_{i,i}|^2 \gamma}{v_i}+1) \\ &+ \frac{1}{\binom{u}{v_{i'}}} \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} v^*|\mathcal{B}_{v^*}| \log(\frac{|h_{i,i}|^2 \gamma}{v_i}+\frac{|h_{i',i}|^2 \gamma}{v_{i'}}+1) \\ &+ \frac{1}{\binom{u}{v_{i'}}} \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} (v_{i'} - v^*)|\mathcal{B}_{v^*}| \log(\frac{|h_{i',i}|^2 \gamma}{v_{i'}}+1) \end{aligned} \tag{67}$$

where $|\mathcal{B}_{v^*}|$ is he cardinality of the set $\mathcal{B}_{v^*}$. As $|\mathcal{B}_{v^*}| = \binom{v_i}{v^*}\binom{u-v_i}{v_{i'}-v^*}$, we have $\frac{|\mathcal{B}_{v^*}|}{\binom{u}{v_{i'}}} = p_{v^*}$. Hence, according to the properties of the hypergeometric distribution stated earlier, we get:

$$\begin{aligned} \mathrm{h}(\vec{Y}_i(s_i^*)) &\geq u \log(2\pi e \sigma^2) \\ &+ \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} (v_i - v^*) p_{v^*} \log(\frac{|h_{i,i}|^2 \gamma}{v_i}+1) \\ &+ \sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k} v^* p_{v^*} \log(\frac{|h_{i,i}|^2 \gamma}{v_i}+\frac{|h_{i',i}|^2 \gamma}{v_{i'}}+1) \end{aligned}$$

$$+\sum_{v^*=(v_0+v_1-u)^+}^{\min_k v_k}(v_{i'}-v^*)p_{v^*}\log(\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)$$

$$=u\log(2\pi e\sigma^2)+(v_i-\frac{v_0v_1}{u})\log(\frac{|h_{i,i}|^2\gamma}{v_i}+1)$$

$$+\frac{v_0v_1}{u}\log(\frac{|h_{i,i}|^2\gamma}{v_i}+\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)+(v_{i'}-\frac{v_0v_1}{u})\log(\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1). \quad (68)$$

• Upper Bound on $\mathrm{h}(\vec{Z}_i)$

Clearly, $\mathrm{h}(\vec{Z}_i)\leq\sum_{j=1}^{u}\mathrm{h}(Z_{i,j})$. On the other hand, based on proposition 1, we have:

$$h(Z_{i,j})\leq\frac{v_{i'}}{u}\log(\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)+\log(2\pi e\sigma^2)+\kappa_i \quad (69)$$

where κ_i is a constant not depending on γ. As a result, we get the following upper bound on $\mathrm{h}(\vec{Z}_i)$:

$$\mathrm{h}(\vec{Z}_i)\leq v_{i'}\log(\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)+u\log(2\pi e\sigma^2)+u\kappa. \quad (70)$$

According to (68) and (70), we deduce the following lower bound on R_i:

$$R_i\geq -u\kappa+(v_i-\frac{v_0v_1}{u})\log(\frac{|h_{i,i}|^2\gamma}{v_i}+1)$$

$$+\frac{v_0v_1}{u}\log(\frac{|h_{i,i}|^2\gamma}{v_i}+\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)+(v_{i'}-\frac{v_0v_1}{u})\log(\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)-v_{i'}\log(\frac{|h_{i',i}|^2\gamma}{v_{i'}}+1)$$

$$=(v_i-\frac{v_0v_1}{u})\log(1+\frac{|h_{i,i}|^2\gamma}{v_i})+\frac{v_0v_1}{u}\log(1+\frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1+\frac{|h_{i',i}|^2\gamma}{v_{i'}}})-u\kappa. \quad (71)$$

Interestingly, the upper and lower bounds derived on R_i are similar up to a deterministic SNR-free difference. Therefore, R_i is asymptotically equivalent to the upper bound derived, i.e.,

$$R_i\sim(v_i-\frac{v_0v_1}{u})\log(1+\frac{|h_{i,i}|^2\gamma}{v_i})+\frac{v_0v_1}{u}\log(1+\frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1+\frac{|h_{i',i}|^2\gamma}{v_{i'}}}). \quad (72)$$

As a result, defining the outage event on the i^{th} link as $\mathcal{O}_i=\{(h_{i,i},h_{i',i}):R_i<r_i\log\gamma\}$, in the high SNR regime, we get:

$$\Pr(\mathcal{O}_i)=\Pr((v_i-\frac{v_0v_1}{u})\log(1+\frac{|h_{i,i}|^2\gamma}{v_i})+\frac{v_0v_1}{u}\log(1+\frac{\frac{|h_{i,i}|^2\gamma}{v_i}}{1+\frac{|h_{i',i}|^2\gamma}{v_{i'}}})<r_i\log\gamma)$$

$$=\Pr((v_i-\frac{v_0v_1}{u})(1-\alpha_{i,i})^++\frac{v_0v_1}{u}(1-\alpha_{i,i}-(1-\alpha_{i',i})^+)^+<r_i). \quad (73)$$

where $r_i \in [0, v_i - \frac{v_0 v_1}{u}]$ as usual. We describe the outage event in different cases.

$\diamondsuit$ $\alpha_{i,i} > 1$ and $\alpha_{i',i} > 1$

Under these conditions, the outage event is equivalent to $r_i > 0$ which is a triviality. Therefore, the outage event in this case is the following:

$$\mathcal{O}_i[1] = \{(\alpha_{i,i}, \alpha_{i',i}) : \alpha_{i,i} > 1, \alpha_{i',i} > 1\}. \tag{74}$$

$\diamondsuit$ $\alpha_{i,i} > 1$ and $\alpha_{i',i} < 1$

It is easily seen that this case is also equivalent to the trivial condition $r_i > 0$. Thus the outage region would be

$$\mathcal{O}_i[2] = \{(\alpha_{i,i}, \alpha_{i',i}) : \alpha_{i,i} > 1, \alpha_{i',i} < 1\}. \tag{75}$$

$\diamondsuit$ $\alpha_{i,i} < 1$ and $\alpha_{i',i} > 1$

We obtain $(v_i - \frac{v_0 v_1}{u})(1 - \alpha_{i,i}) + \frac{v_0 v_1}{u}(1 - \alpha_{i,i}) = v_i(1 - \alpha_{i,i}) < r_i$. Denoting the outage region in this case by $\mathcal{O}_i[3]$, we have:

$$\mathcal{O}_i[3] = \{(\alpha_{i,i}, \alpha_{i',i}) : \alpha_{i,i} < 1, \alpha_{i',i} > 1, v_i(1 - \alpha_{i,i}) < r_i\}. \tag{76}$$

$\diamondsuit$ $\alpha_{i,i} < 1$ and $\alpha_{i',i} < 1$

We get $(y_i - \frac{v_0 v_1}{u})(1 - \alpha_{i,i}) + \frac{v_0 v_1}{u}(\alpha_{i',i} - \alpha_{i,i})^+ < r_i$. If $\alpha_{i,i} < \alpha_{i',i}$, then we get the following region:

$$\mathcal{O}_i[4] = \left\{(\alpha_{i,i}, \alpha_{i',i}) : \begin{array}{c} \alpha_{i,i} < 1, \alpha_{i',i} < 1, \alpha_{i,i} < \alpha_{i',i} \\ v_i - \frac{v_0 v_1}{u} - v_i \alpha_{i,i} + \frac{v_0 v_1}{u} \alpha_{i',i} < r_i \end{array} \right\}. \tag{77}$$

If $\alpha_{i,i} > \alpha_{i',i}$, the outage region is the following set:

$$\mathcal{O}_i[5] = \{(\alpha_{i,i}, \alpha_{i',i}) : \alpha_{i,i} < 1, \alpha_{i',i} < 1, \alpha_{i,i} > \alpha_{i',i}, (v_i - \frac{v_0 v_1}{u})(1 - \alpha_{i,i}) < r_i\}. \tag{78}$$

We have $\mathcal{O}_i = \cup_{l=1}^{5} \mathcal{O}_i[l]$ where $\mathcal{O}_i[l] \cap \mathcal{O}_i[l'] = \varnothing$ for $l \neq l'$. Thus,

$$\Pr(\mathcal{O}_i) = \sum_{l=1}^{5} \Pr(\mathcal{O}_i[l]). \tag{79}$$

If we set $\Pr(\mathcal{O}_i[l]) \doteq \gamma^{-d_i[l]}$, then clearly $d_i = \min_l d_i[l]$ where $\Pr(\mathcal{O}_i) \doteq \gamma^{d_i}$. In the following we obtain $d_i[l] = \min_{(t_1,t_2) \in \mathcal{O}_i[l]} t_1 + t_2$.

$\diamondsuit$ $l = 1, 2, 3$ The outage region has a simple structure, and one can simply verify that $d_i[1] = 2$,

$d_i[2] = 1$ and $d_i[3] = (1 - \frac{r_i}{v_i})^+$.

$\lozenge$ $l = 4$

Fig. 1 illustrates $\mathcal{O}_i[4]$. We simply get $d_i[4] = 2(1 - \frac{r_i}{v_i a_{i,0}})^+$ where $a_{i,0} = 1 - \frac{v_{i'}}{u}$.

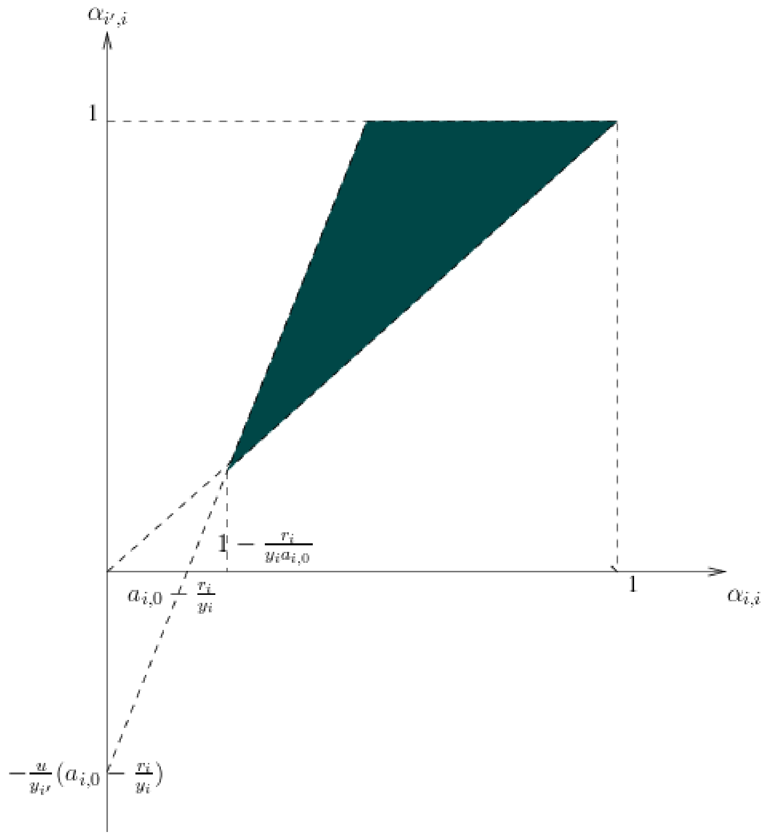

Fig. 1. The Region $\mathcal{O}_i[4]$

$\lozenge$ $l = 5$

Fig. 2 depicts the outage region $\mathcal{O}_i[5]$. Based on this sketch, $d_i[5] = (1 - \frac{r_i}{v_i a_{i,0}})^+$.

Now, one can easily see $d_i = \min_l d_i[l] = d_i[5] = (1 - \frac{r_i}{v_i a_{i,0}})^+$. This is the desired result.

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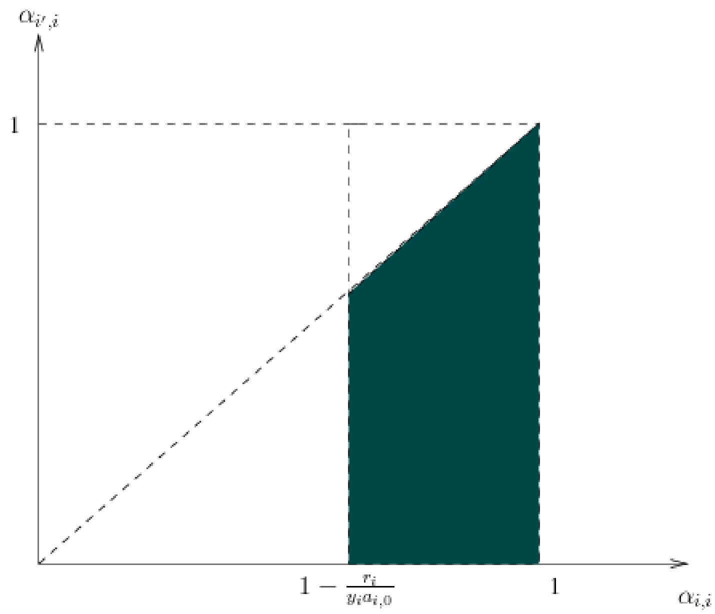

Fig. 2. The Region $\mathcal{O}_i[5]$

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