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in The Multiple Relays Network**

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# Optimum Diversity-Multiplexing Tradeoff in The Multiple Relays Network

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## I. SYSTEM MODEL

The system, as in [1], [2], and [3], consists of  $K$  relays assisting the transmitter and the receiver in the half-duplex mode, i.e. in each time, the relays can either transmit or receive. The channels between each two node is assumed to be quasi-static flat Rayleigh-fading, i.e. the channel gains remain constant during a block of transmission and changes independently from one block to another. However, we assume that there is no direct link between the transmitter and the receiver. This assumption is reasonable when the transmitter and the receiver are far from each other or when the receiver is supposed to have connection with just the relay nodes to avoid the complexity of the network. As in [2] and [4], each node is assumed to know the state of its backward channel and, moreover, the receiver is supposed to know the equivalent channel gain from the transmitter to the receiver. No feedback to the transmitting node is permitted. All nodes have the same power constraint. Also, we assume that a capacity achieving gaussian random codebook can be generated at each node of the network. Hence, the code design problem is not considered in this paper.

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## II. PROPOSED $K$ -SLOT SWITCHING $N$ -SUB-BLOCK MARKOVIAN SCHEME (SM)

In the proposed scheme, the entire block of transmission is divided into  $N$  sub-blocks. Each sub-block consists of  $K$  slots. Each slot has  $T'$  symbols. Hence, the entire block consists of  $T = NKT'$  symbols. In order to transmit a message  $w$ , the transmitter selects the corresponding codeword of a gaussian random codebook consisting of  $2^{NKT'r}$  codewords of length  $\frac{NK-1}{NK}T$  and transmits the codeword during the first  $NK - 1$  slots. In each sub-block, each relay receives the signal in one of the slots and transmits the received signal in the next slot. So, each relay is off in  $\frac{K-2}{2}$  of time. More precisely, in the  $k'$  slot of the  $n'$ th sub-block ( $1 \leq n \leq N, 1 \leq k \leq K, nk \neq NK$ ), the  $k'$ th relay receives the signals the transmitter is sending, and amplifies and forwards it to the receiver in the next slot. The receiver starts receiving the signal from the second slot. After receiving the last slot ( $NK'$ th slot) signal, the receiver decodes the transmitted message by using the signal of  $NK - 1$  slot received from  $K$  relays. It will be shown in the next section that the equivalent point-to-point channel from the transmitter to the receiver would act as a lower-triangular MIMO channel.

## III. DIVERSITY-MULTIPLEXING TRADEOFF

In this section, we show that the proposed method achieves the optimum achievable diversity-multiplexing curve. First, according to the cut-set bound theorem [5], the point-to-point capacity of the uplink channel (the channel from the transmitter to the relays) is an upper-bound for the capacity of this system. Accordingly, the diversity-multiplexing curve of a  $1 \times K$  SIMO system which is a straight line from multiplexing gain 1 to the diversity gain  $K$  is an upper-bound for the diversity-multiplexing curve of our system. In this section, we prove that the tradeoff curve of the proposed method achieves the upper-bound and thus, it is optimum. First, we prove the statement for the case that there is no link between the relays. Next, we prove the statement for the general case.

### A. No Interfering Relays

Assume, the link gain between the  $k'$ th relay and the transmitter and the  $k'$ th relay and the receiver are  $h_k$  and  $g_k$ , respectively. Furthermore, assume that there is no link between the relays. Accordingly, at the  $k'$ th relay we have

$$\mathbf{r}_k = h_k \mathbf{x} + \mathbf{n}_k, \quad (1)$$

where  $\mathbf{r}_k$  is the received signal vector of the  $k$ 'th relay,  $\mathbf{x}$  is the transmitter signal vector and  $\mathbf{n}_k \sim \mathcal{N}(0, \mathbf{I}_{T'})$  is the noise vector of the channel. At the receiver side, we have

$$\mathbf{y} = \sum_{k=1}^K g_k \mathbf{t}_k + \mathbf{z}, \quad (2)$$

where  $\mathbf{t}_k$  is the transmitted signal vector of the  $k$ 'th relay,  $\mathbf{y}$  is the received signal vector at the receiver side and  $\mathbf{z} \sim \mathcal{N}(0, \mathbf{I}_{T'})$  is the noise vector of the downlink channel. The output power constraint  $\mathbb{E} \{ \|\mathbf{x}\|^2 \}, \mathbb{E} \{ \|\mathbf{t}_k\|^2 \} \leq T'P$  holds at the transmitter and relays side. To obtain the DM tradeoff curve of the proposed scheme, we are looking for the end-to-end probability of outage from the rate  $r \log(P)$ , as  $P$  goes to infinity.

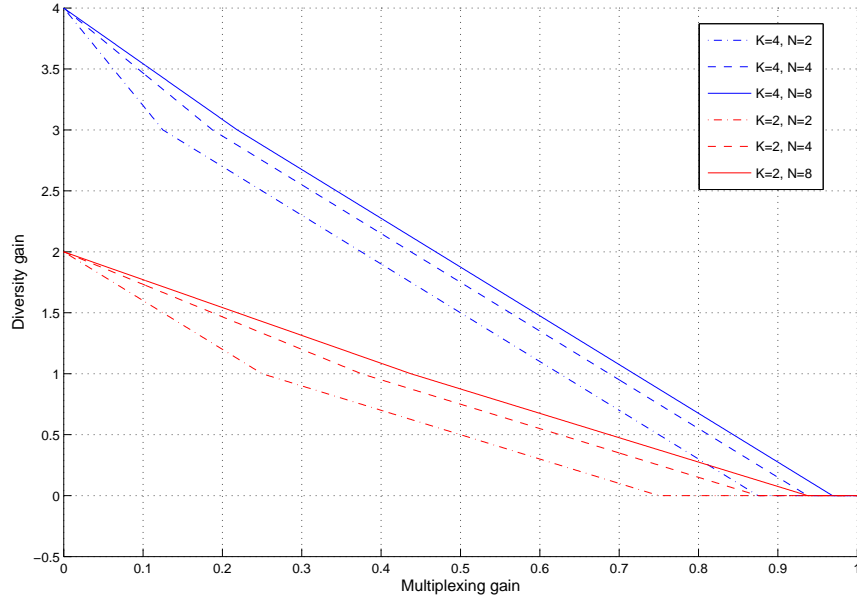


Fig. 1. DM Tradeoff for the proposed Switching Markovian Scheme and various values of (K,N), No interfering relays case

**Theorem 1** Assume a half-duplex parallel relay scenario with  $K$  no interfering relays. The proposed SM scheme achieves the diversity gain

$$d_{SM,NI}(r) = \max \left\{ 0, K(1-r) - \frac{1}{N}, K(1-r) - \frac{Kr}{N-1} \right\}, \quad (3)$$

which achieves the optimum achievable DM tradeoff curve  $d_{opt}(r) = K(1-r)$  as  $N \rightarrow \infty$ .

*Proof:* Let us define  $\mathbf{x}_{n,k}$ ,  $\mathbf{n}_{n,k}$ ,  $\mathbf{r}_{n,k}$ ,  $\mathbf{t}_{n,k}$ ,  $\mathbf{z}_{n,k}$ ,  $\mathbf{y}_{n,k}$  as the signal/noise transmitted/received by the transmitter/relay/receiver to the  $k$ 'th relay/receiver in the  $k$ 'th slot of the  $n$ 'th sub-block. Also, let us define  $(k) \equiv k - 2 \pmod{K + 1}$  and  $(n) \equiv n - \lfloor \frac{k}{K} \rfloor$ . Thus, we have

$$\begin{aligned} \mathbf{y}_{n,k} &= g_k \mathbf{t}_{n,k} + \mathbf{z}_{n,k} \\ &= g_k \alpha_{(k)} \left( h_{(k)} \mathbf{x}_{(n),(k)} + \mathbf{n}_{(n),(k)} \right) + \mathbf{z}_{n,k}, \end{aligned} \quad (4)$$

where  $\alpha_k = \frac{P}{|h_k|^2 P + 1}$  is the amplification coefficient performed in the  $k$ 'th relay. Defining the event  $\mathcal{E}_k$  as the event of outage from the rate  $r \log(P)$  in the  $k$ 'th sub-channel consisting of the transmitter, the  $k$ 'th relay, and the receiver, we have

$$\begin{aligned} \mathbb{P}\{\mathcal{E}_k\} &= \mathbb{P} \left\{ \log \left[ 1 + P |g_k|^2 |\alpha_k|^2 |h_k|^2 (1 + |g_k|^2 |\alpha_k|^2)^{-1} \right] \leq r \log(P) \right\} \\ &\doteq \min \left\{ \text{sign}(r), \mathbb{P} \left\{ |g_k|^2 |\alpha_k|^2 |h_k|^2 (1 + |g_k|^2 |\alpha_k|^2)^{-1} \leq P^{r-1} \right\} \right\} \\ &\stackrel{(a)}{\doteq} \min \left\{ \text{sign}(r), \mathbb{P} \left\{ |g_k|^2 |\alpha_k|^2 |h_k|^2 \min \left\{ \frac{1}{2}, \frac{1}{2|g_k|^2 |\alpha_k|^2} \right\} \leq P^{r-1} \right\} \right\} \\ &\stackrel{(b)}{\doteq} \min \left\{ \text{sign}(r), \mathbb{P} \left\{ |h_k|^2 \leq 2P^{r-1} \right\} + \mathbb{P} \left\{ |g_k|^2 |\alpha_k|^2 |h_k|^2 \leq 2P^{r-1} \right\} \right\} \\ &\stackrel{(c)}{\doteq} \min \left\{ \text{sign}(r), P^{-(1-r)} + \mathbb{P} \left\{ |g_k|^2 \min \left\{ \frac{1}{2}, \frac{|h_k|^2 P}{2} \right\} \leq 2P^{r-1} \right\} \right\} \\ &\stackrel{(d)}{\doteq} \min \left\{ \text{sign}(r), P^{-(1-r)} + \mathbb{P} \left\{ |g_k|^2 \leq 4P^{r-1} \right\} + \mathbb{P} \left\{ |g_k|^2 |h_k|^2 \leq 4P^{r-2} \right\} \right\} \\ &\stackrel{(e)}{\doteq} \min \left\{ \text{sign}(r), P^{-(1-r)} \right\}, \end{aligned} \quad (5)$$

where  $\text{sign}(r)$  is the sign function, i.e.  $\text{sign}(r) = 1, r \geq 0, \text{sign}(r) = 0, r < 0$ . Here, (a) follows from the fact that  $\frac{1}{1+|g_k|^2|\alpha_k|^2} \doteq \min \left\{ \frac{1}{2}, \frac{1}{2|g_k|^2|\alpha_k|^2} \right\}$ , (b) and (d) follow from the union bound inequality, (c) follows from the fact that  $|\alpha_k|^2 |h_k|^2 \doteq \min \left\{ \frac{1}{2}, \frac{|h_k|^2 P}{2} \right\}$  and the pdf distribution of the rayleigh-fading parameter near zero, and (e) follows from the fact that the product of two independent rayleigh-fading parameters behave as a rayleigh-fading parameter near zero. (5) shows that each sub-channel's tradeoff curve performs as a single-antenna point-to-point channel.

Defining  $R_k(P)$  as the random variable showing the rate of the  $k$ 'th sub-channel consisting of the transmitter, the  $k$ 'th relay, and the receiver in terms of  $P$ , the outage event of the entire

channel from the  $r \log(P)$ , the event  $\mathcal{E}$ , is equal to

$$\mathbb{P}\{\mathcal{E}\} = \mathbb{P}\left\{N \sum_{k=1}^{K-1} R_k(P) + (N-1)R_K(P) \leq NKr \log(P)\right\} \quad (6)$$

Assuming  $R_k(P) = r_k \log(P)$ , we have

$$\mathbb{P}\{\mathcal{E}\} \doteq \mathbb{P}\left\{N \sum_{k=1}^{K-1} r_k + (N-1)r_K \leq NKr\right\} \quad (7)$$

$\mathbb{P}\{R_k(P) \leq r_k \log(P)\}$  is known by (5). Defining the region  $\mathcal{R}$  as

$$\mathcal{R} = \left\{(r_1, r_2, \dots, r_K) \mid 0 \leq r_k \leq 1, N \sum_{k=1}^{K-1} r_k + (N-1)r_K \leq NKr\right\} \quad (8)$$

it is easy to check that all the vectors  $(r_1, r_2, \dots, r_K)$  that result in the outage event almost surely lie in  $\mathcal{R}$ . In fact, according to (5), for all  $k$  we know  $r_k \geq 0$ . Also, for  $r_k > 1$ ,  $\mathbb{P}\{R_k(P) \geq r_k \log(P)\} \leq e^{-Pr^{-1}}$  which is exponential in terms of  $P$ . Hence,  $r_k > 1$  can be disregarded for the outage region. As a result,  $\mathbb{P}\{\mathcal{E}\} \doteq \mathbb{P}\{\mathbf{r} \in \mathcal{R}\}$ .

On the other hand, by (5) and the fact that  $r_k$ 's are independent, we have

$$\mathbb{P}\{r_1 \leq r_1^0, r_2 \leq r_2^0, \dots, r_K \leq r_K^0\} \doteq P^{-(K - \sum_{k=1}^K r_k^0)} \quad (9)$$

Now, we show that  $\mathbb{P}\{\mathcal{E}\} \doteq P^{-\min_{\mathbf{r} \in \mathcal{R}} K-1 \cdot \mathbf{r}}$ . First of all, by taking derivative of (9) with respect to  $r_1, r_2, \dots, r_K$ , it is easy to see that the probability density function of  $\mathbf{r}$  behaves the same as the probability function in (9), i.e.  $f_r(\mathbf{r}) \doteq P^{-(K-1 \cdot \mathbf{r})}$ . Hence, the outage probability is equal to

$$\begin{aligned} \mathbb{P}\{\mathcal{E}\} &\doteq \int_{\mathbf{r} \in \mathcal{R}} f_r(\mathbf{r}) d\mathbf{r} \\ &\leq \text{vol}(\mathcal{R}) P^{-\min_{\mathbf{r} \in \mathcal{R}} K-1 \cdot \mathbf{r}} \\ &\stackrel{(a)}{\doteq} P^{-\min_{\mathbf{r} \in \mathcal{R}} K-1 \cdot \mathbf{r}} \end{aligned} \quad (10)$$

Here, (a) follows from the fact that  $\mathcal{R}$  is a fixed bounded region whose volume is independent of  $P$ . On the other hand, by continuity of  $P^{-(K-1 \cdot \mathbf{r})}$  over  $\mathbf{r}$ , we have  $\mathbb{P}\{\mathcal{E}\} \doteq P^{-\min_{\mathbf{r} \in \mathcal{R}} K-1 \cdot \mathbf{r}}$  which combining with (10), results into  $\mathbb{P}\{\mathcal{E}\} \doteq P^{-\min_{\mathbf{r} \in \mathcal{R}} K-1 \cdot \mathbf{r}}$ . Defining  $l(\mathbf{r}) = K-1 \cdot \mathbf{r}$ , we have to solve the following linear programming optimization problem  $\min_{\mathbf{r} \in \mathcal{R}} l(\mathbf{r})$ . Notice that the region  $\mathcal{R}$  is defined by a set of linear inequality constraints. To solve the problem, we have

$$\begin{aligned} l(\mathbf{r}) &\stackrel{(a)}{\geq} \max \left\{ 0, K - \frac{NKr + r_K}{N}, K - \frac{NKr - \sum_{k=1}^{K-1} r_k}{N-1} \right\} \\ &\stackrel{(b)}{\geq} \max \left\{ 0, K(1-r) - \frac{1}{N}, K(1-r) - \frac{Kr}{N-1} \right\}. \end{aligned} \quad (11)$$

Here, (a) follows from the inequality constraint in (8) governing  $\mathcal{R}$ , and (b) follows from the fact that  $r_K \leq 1$  and  $\forall k < K : r_k \geq 0$ . Now, we partition the range  $0 \leq r \leq 1$  into three intervals. First, in the case that  $r > 1 - \frac{1}{NK}$ , the feasible point  $\mathbf{r} = \mathbf{1}$  achieves the lower bound 0. Second, in the case that  $r < \frac{1}{K} - \frac{1}{NK}$ , the feasible point  $\mathbf{r} = (0, 0, \dots, 0, \frac{NKr}{N-1})$ , achieves the lower bound  $K(1-r) - \frac{Kr}{N-1}$ . Finally, in the case that  $\frac{1}{K} - \frac{1}{NK} \leq r \leq 1 - \frac{1}{NK}$ , The lower bound  $K(1-r) - \frac{1}{N}$  is achievable by the feasible point  $\mathbf{r}, \forall k < K : r_k = \frac{NKr-N+1}{N(K-1)}, r_K = 1$ . Hence, we have  $\min_{\mathbf{r} \in \mathcal{R}} l(\mathbf{r}) = \max \{0, K(1-r) - \frac{1}{N}, K(1-r) - \frac{Kr}{N-1}\}$ . This completes the proof. ■

*Remark* - It is worth noting that as long as the graph  $G(V, E)$  whose vertices are the relay nodes and edges are the non interfering relay node pairs includes a hamiltonian cycle<sup>1</sup>, the result of this subsection remains valid.

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<sup>1</sup>By hamiltonian cycle, we mean a simple cycle  $v_1 v_2 \dots v_K v_1$  that goes exactly one time through each vertex of the graph.