Path Diversity over the Internet: Performance Analysis and Rate Allocation

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Abstract

Path diversity works by setting up multiple parallel connections between the end points using the topological path redundancy of the network. In this paper, *Forward Error Correction* (FEC) is applied across multiple independent paths to enhance the end-to-end reliability. Internet paths are modeled as erasure Gilbert-Elliot channels [1], [2]. First, it is shown that over any erasure channel, *Maximum Distance Separable* (MDS) codes achieve the minimum probability of irrecoverable loss among all block codes of the same size. Then, based on the adopted model for Internet paths, we prove the probability of irrecoverable loss for MDS codes decays exponentially for the asymptotically large number of paths. Moreover, it is shown that in the optimal rate allocation, each path is assigned a positive rate *iff* its quality is above a certain threshold. The quality of a path is defined as the percentage of the time it spends in the bad state. In other words, including a redundant path improves the reliability *iff* this condition is satisfied. Finally, using dynamic programming, a heuristic suboptimal algorithm with polynomial runtime is proposed for rate allocation over the available paths. This algorithm converges to the aymptotically optimal rate allocation when the number of paths is large. The simulation results show that the proposed algorithm approximates the optimal rate allocation very closely, and provides significant performance improvement compared to the alternative schemes of rate allocation.

I. Introduction

In recent years, *path diversity* over the Internet has received significant attention. It has been shown that path diversity has the ability to simultaneously improve the end-to-end rate and reliability [3]–[5]. In a dense network like the Internet, it is usually possible to find multiple disjoint paths between any pair of nodes [6]. In this paper, *Forward Error Correction* (FEC) is applied across multiple disjoint paths. Knowing that packet loss and delay patterns are independent over such paths, we expect path diversity to enhance the performance of FEC.

References [7] and [8] have proposed both centralized and distributed algorithms to find multiple independent paths over a large connected graph. Although the distributed algorithms do not require the end nodes to know the entire topology of the network, they impose some modifications on the intermediate nodes. Indeed, modification of the routing protocol and signaling between the nodes is extremely costly over the traditional IP networks. To avoid such an expense, overlay networks are introduced [9]. The basic idea of overlay networks is to equip a few number of nodes (smart nodes) with the desired new functionalities while the rest remain unchanged. The smart nodes form a virtual network connected through virtual or logical links on top of the actual network. Thus, the overlay networks can be used to set up disjoint paths between the end nodes.

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Reference [10] addresses the problem of overlay network design based on a game theoretical approach. Also, reference [6] has experimentally studied the number of available disjoint paths in the Internet using overlay networks.

Recently, path diversity is utilized in many applications (see [2], [11], [12]). Reference [12] combines multiple description coding and path diversity to improve quality of service (QoS) in video streaming. Packet scheduling over multiple paths is addressed in [13] to optimize the rate-distortion function of a video stream. In [3], multipath routing of TCP packets is applied to control the congestion with minimum signaling overhead. Moreover, references [5] and [4] study the problem of rate allocation over multiple paths. Assuming each path follows the leaky bucket model, reference [5] shows that a water-filling based scheme provides the minimum end-to-end delay.

On the other hand, reference [4] considers a scenario of multiple senders and a single receiver, assuming all the senders share the same source of data. The connection between each sender and the receiver is assumed to follow the Gilbert-Elliot model. They propose a receiver-driven protocol for packet partitioning and rate allocation. The packet partitioning algorithm ensures no sender sends the same packet, while the rate allocation algorithm minimizes the probability of irrecoverable loss in the FEC scheme [4]. They apply a brute-force search algorithm to solve the rate allocation problem over two paths only. However, it should be noted that the scenario of [4] can be simplified, without any loss of generality, into the case of multiple independent paths between a pair of end nodes as the senders share the same data.

Maximum Distance Separable (MDS) codes has been shown to be optimum in the sense that they achieve the maximum possible minimum distance (d_{min}) among block codes of the same size [14]. Indeed, any $[n, k]$ MDS code with block length n and k symbols of informatoin can be successfully recovered from any subset of its entries of length k or more. This property makes MDS codes favorable FEC schemes over the erasure channels like the Internet [15]–[17]. However, the simple and practical encoding-decoding algorithms for such codes have quadratic time complexity in terms of the code size [18]. Theoretically, more efficient ($O\left(n \log^2(n)\right)$) MDS codes can be constructed based on evaluating and interpolating polynomials over specially chosen finite fields using Discrete Fourier Transform [19], but these methods are not competitive in practice with the simpler quadratic methods except for extremely large block sizes. Recently, a family of almost-MDS codes with low encoding-decoding time complexity (linear in term of the code length) is proposed and shown to be practical over the erasure channels like the Internet [20], [21]. In these codes, any subset of symbols of size $k(1+\epsilon)$ is sufficient to recover the original k symbols with high probability [21]. Furthermore, MDS codes suffer from having large alphabet sizes. Indeed, all the known MDS codes have alphabet sizes growing at least linearly with the block length n. There is a conjecture stating that all the $[n, k]$ MDS codes with $1 < k < n-1$ have the property that $n \leq q+1$ with two exceptions [14]. However, this is not an issue in the practical networking applications since the alphabet size is 2^r where r is the packet size, i.e. the block size is much smaller than the alphabet size.

In this work, we utilize path diversity to improve the performance of FEC. The details of path setup process is not discussed here. It is just assumed that L independent paths are set up by a smart overlay network or any other means. Probability of irrecoverable loss (P_E) is defined as the measure of FEC performance. First, it is shown that MDS block codes have the minimum probability of error over our *Internet Channel* model, and over any other erasure channel with or without memory. Applying MDS codes in FEC, our analysis shows an exponential decay of P_E with respect to the number of paths. Of course, in many practical cases, the number of *independent* or *disjoint* paths between the end nodes is limitted. However, in our asymptotic analysis, we have assumed that it is possible to find L independent paths between the end points even when L is large. Moreover, the optimal rate allocation problem is solved in the asymptotic case where the number of paths becomes very large. It is seen that in the optimal rate allocation, each path is assigned a positive rate *iff* its quality is above a certain threshold.

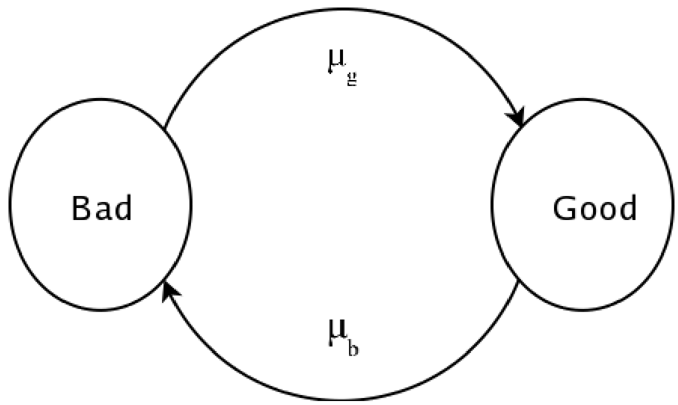


Fig. 1. Continuous-time two-state Markov model of an Internet channel

Modeling each path with a two-state continuous time markov process called Gilbert-Elliot channel [1], [2], the quality of a path is defined as the percentage of the time it spends in the bad state. Furthermore, using dynamic programming, a heuristic suboptimal algorithm is proposed for rate allocation over the available paths. Unlike the brute-force search, this algorithm has a polynomial complexity, in terms of the number of paths. It is shown that the result of this algorithm converges to the asymptotically optimal solution for large number of paths. Finally, the simulation results verify the near-optimal performance of the proposed algorithm.

The rest of this paper is organized as follows. Section II describes the system model. Probability distribution of the bad burst duration is discussed in section III. Performance of FEC in three cases of a single path, multiple identical paths, and non-identical paths are analyzed in section IV. Section V studies the rate allocation problem, and proposes a suboptimal rate allocation algorithm. Finally, section VI concludes the paper.

II. System Modeling and Formulation

A. Internet Channel Model

From an end to end protocol's perspective, performance of the lower layers in the protocol stack can be modeled as a random *channel* called *Internet channel*. Since each packet usually includes an internal error detection coding (for instance a CRC), the Internet channel is satisfactorily modeled as an erasure channel. Although there is no well defined capacity limit for the Internet channel, a maximum TCP-friendly rate is introduced in [22]. Delay of an Internet channel is strongly correlated with the packet loss pattern, and affects the QoS considerably [23], [24].

In this work, the model assumed for the Internet channel is a two-state Markov model called Gilbert-Elliot cell, depicted in Fig. 1. The channel spends an exponentially distributed random amount of time with the mean $\frac{1}{\mu_g}$ in the *Good* state. Then, it alternates to the *Bad* state and stays in that state for another random duration exponentially distributed with the mean $\frac{1}{\mu_b}$. If a packet is transmitted from the source at anytime during the good state, it will be received correctly. Otherwise, if it is transmitted during the bad state, it will eventually be lost before reaching the destination. Therefore, the average probability of error is equal to the steady state probability of being in the bad state $\pi_b = \frac{\mu_g}{\mu_g+\mu_b}$. To have a reasonably low probability of error, μ_g must be much smaller than μ_b. It is also assumed the channel state does not change during the transmission of one packet [2]. This model is widely used for theoretical analysis where delay is not a significant factor [1]. Despite its simplicity, this model captures the bursty error characteristic of Internet channels satisfactorily. More comprehensive models like hidden

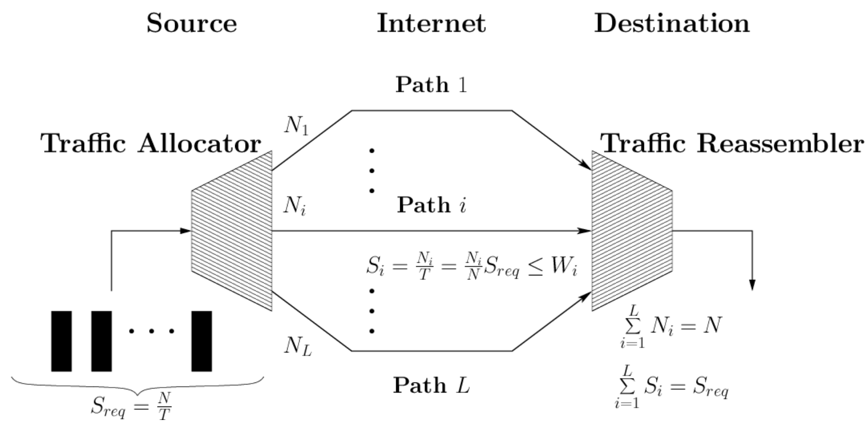


Fig. 2. Rate allocation problem: a block of N packets is being sent from the source to the destination through L independent paths over the Internet during the time Interval T with the required rate $S_{req} = \dfrac{N}{T}$. The block is distributed over the paths according to the vector $\mathbf{N} = (N_1, \ldots, N_L)$ which corresponds to the rate allocation vector $\mathbf{S}$

Markov models are introduced in [24]. Although analytically cumbersome, such models express the correlation of loss and delay more accurately.

B. Typical FEC Model

A concatenated coding is used for packet transmission. The coding inside each packet can be a simple CRC which enables the receiver to detect an error inside each packet. Then, the receiver can consider the Internet channel as an erasure channel. Other than the coding inside each packet, a *Forward Error Correction* (FEC) scheme is applied between packets. Every K packets are encoded to a *Block* of N packets where $N > K$ to create some redundancy. The ratio of $\alpha = \frac{N-K}{N}$ defines the FEC overhead. A *Maximum Distance Separable* (MDS) (N, K) code, such as Reed-Solomon code, can reconstruct the original K data packets on the receiver side if K or more of the N packets are received correctly [14]. According to the following proposition, a MDS code is the optimum block code we can design over any erasure channel. Although FEC imposes some bandwidth overhead, it might be the only option when feedback and retransmission are not feasible or fast enough to provide the desirable QoS.

Proposition I. An erasure channel is defined as the one which maps every input symbol to either itself or an erasure symbol ξ. Consider an arbitrary erasure channel (memoryless or with memory) with the input vector $\mathbf{x} \in \mathcal{X}^N$, $|\mathcal{X}|$, the output vector $\mathbf{y} \in (\mathcal{X} \cup \{\xi\})^N$, and the transition probability $p(\mathbf{y}|\mathbf{x})$ satisfying:

1) $p\left(y_j \notin \{x_j, \xi\} | \, x_j\right) = 0$.
2) $p(\mathbf{y}|\mathbf{x})$ is independent of the input vector $\mathbf{x}$ if $\forall j \;\, y_j \in \{x_j, \xi\}$.

A block code (N, K) with equiprobable codewords over this channel has the minimum probability of error using the optimum (maximum likelihood) decoder among all block codes of the same size *iff* that code is *Maximum Distance Separable* (MDS). The proof can be found in the appendix A.

C. Rate Allocation Problem

Let us assume all packets have the equal length of r bits. Then, all rates can be expressed in pkt/sec instead of bps. From source to the destination, there exist L independent paths with maximum rates of W_i each, as indicated in Fig. 2. W_i can be

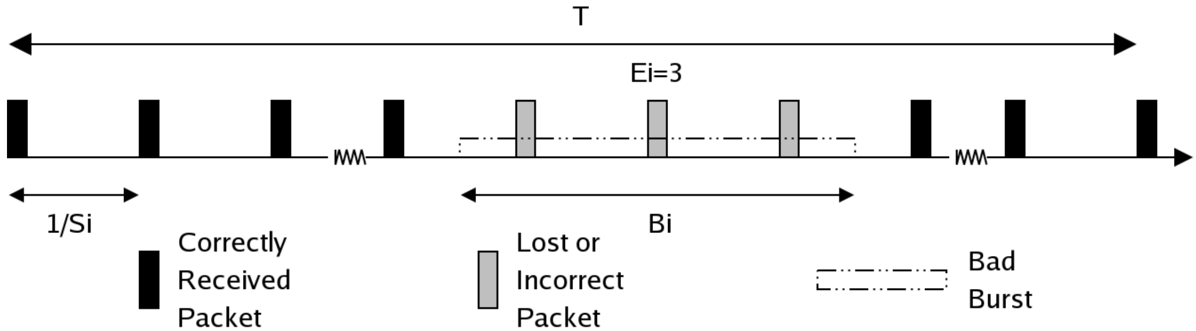


Fig. 3. A bad burst of duration B happens in a block of length T. $E = 3$ packets are corrupted or lost during the interval B. Packets are transmitted every $\frac{1}{S}$ seconds, where S is the rate in pkt/sec.

considered as the maximum TCP-friendly bandwidth introduced in reference [22] or any other upperbound imposed by the physical characteristics of the i'th path. In any case, W_i is assumed to be given for each path in our analysis. For a specific application and FEC scheme, we require the rate S_{req} from source to the destination. Obviously, we should have $S_{req} \leq \Sigma W_i$ to have a feasible solution.

The total time to send a block of N packets is $T = \frac{N}{S_{req}}$, and has an important role in our analysis. The block length is typically much larger than the number of paths ($N \gg L$). According to the FEC model, we can send N_i packets through the path i as long as $\sum N_i = N$ and $\frac{N_i}{T} \leq W_i$. Then, the rate assigned to path i can be expressed as $S_i = \frac{N_i}{T} = \frac{N_i}{N} S_{req}$. Obviously, we have $\sum S_i = S_{req}$. The objective of the rate allocation problem is to find the optimal rate allocation vector or the vector $\mathbf{N} = (N_1, \cdots, N_L)$ which minimizes the probability of irrecoverable loss (P_E).

In this work, an irrecoverable loss is defined as the event where more than K packets are lost in a block of N packets. It should be noted that this definition is slightly different from decoding error in a block code of size (N, K). Theoretically, an optimum maximum likelihood decoder of a MDS code may still decode the original codeword correctly with a positive but very small probability if it receives less than K symbols (packets). For instance, such a decoder for a MDS code over $\mathbb{F}_q$ can decode correctly with the probability of $\frac{1}{q}$ after receiving $K - 1$ correct symbols. However, for Galois fields with large cardinality, this probability is negligible (see the proof of Proposition I in the appendix for more details). Moreover, while many practical low-complexity decoders of MDS codebooks work perfectly if the number of correctly received symbols is at least K, their probability of correct decoding is much less than that of maximum likelihood decoders when the number of correctly received symbols is less than K. Hence, in the rest of this paper, P_E is used as a close approximation of decoding error.

III. Probability Distribution of Bad Bursts

The continuous random variable B_i is defined as the duration of time that the path i spends in the bad state in a block duration, T. We denote the values of B_i with parameter t to emphasize that they are expressed in the time unit. In this section, we focus on one path, for example path 1. Therefore, the index i can be temporarily dropped in analyzing the distribution function of B_i.

We define the events g and b, respectively as the channel being in the good or bad states at the start of a block. Then, the distribution of B can be written as

$$f_B(t) = f_{B|b}(t)\pi_b + f_{B|g}\pi_g. \tag{1}$$

To proceed further, two assumptions are made. First, it is assumed that $\pi_g \gg \pi_b$ or equivalently $\frac{1}{\mu_g} \gg \frac{1}{\mu_b}$. This condition is valid for a channel with a reasonable quality. Besides, the block time T is assumed to be much shorter than the average good state duration $\frac{1}{\mu_g}$, i.e. $1 \gg \mu_g T$, such that the interval T can contain only zero or one intervals of bad burst (see [1], [2], [4] for justification).

Hence, the distribution of B conditioned on the event b is approximated as

$$f_{B|b}(t) \approx \mu_b e^{-\mu_b t} + \delta(t-T)e^{-\mu_b T} \tag{2}$$

where $\delta(u)$ is the Dirac delta function. (2) follows from the memoryless nature of the exponential distribution, the assumption that T contains at most one bad burst, and the fact that any bad burst longer than T has to be truncated at $B = T$.

To approximate $f_{B|g}(t)$, we have

$$f_{B|g}(t) = \mathbb{P}\{B = 0|g\}\delta(t) - \frac{\partial}{\partial t}\mathbb{P}\{B > t|g\} \tag{3}$$

where

$$\mathbb{P}\{B = 0|g\} = e^{-\mu_g T} \approx 1 - \mu_g T \tag{4}$$

and

$$\mathbb{P}\{B > t|g\} \overset{(a)}{\approx} (1 - e^{-\mu_g(T-t)})e^{-\mu_b t} \approx \mu_g(T-t)e^{-\mu_b t} \tag{5}$$

where (a) is resulted from the fact that $\{B > t|g\}$ is equivalent to the initial good burst being shorter than $T - t$ and the following bad burst larger than t. Now, combining (4), (5), (3), and (2), $f_B(t)$ can be computed.

A. The Expected Value of B

Throughout this paper, $\mathbb{E}\{\}$ stands for the expected value operator. In order to find $\mathbb{E}\{B\}$ based on its distribution, we write

$$\begin{aligned}
\mathbb{E}\{B\} &= \mathbb{E}\{B|g\}\pi_g + \mathbb{E}\{B|b\}\pi_b \\
\mathbb{E}\{B|b\} &= \int_0^T t f_{B|b}(t)dt = \frac{1}{\mu_b}(1 - e^{-\mu_b T}) \overset{(a)}{\approx} \frac{1}{\mu_b} \\
\mathbb{E}\{B|g\} &= \int_0^T t f_{B|b}(t)dt \\
&= \frac{\mu_g(\mu_b T + 3)}{\mu_b^2} - \frac{\mu_g T e^{-\mu_b T}(\mu_b T + 3)}{\mu_b} - \frac{\mu_g e^{-\mu_b T}(\mu_b T + 3)}{\mu_b^2} - \mu_g T^2 e^{-\mu_b T} \\
&\overset{(b)}{\approx} \frac{\mu_g}{\mu_b} T \approx \pi_b T
\end{aligned} \tag{6}$$

where both (a) and (b) follow from $\mu_g << \mu_b$ and $T >> \frac{1}{\mu_b}$, assuming the block transmission time being much larger than the average bad burst length. Accordingly, $\mathbb{E}\{B\}$ can be approximated as

$$\mathbb{E}\{B\} = \pi_b T \pi_g + \frac{1}{\mu_b}\pi_b \approx \pi_b T \tag{7}$$

B. Discrete to Continuous Approximation

To detect an irrecoverable loss, we are interested in the probability of k_i packets being lost out of the N_i packets transmitted through the path i. Let us denote the number of erroneous or lost packets with the random variable E_i. Any two subsequent packets in a block are $\frac{1}{S_i}$ seconds apart in time, where S_i is the transmission rate over the path i. Assuming there can be at most one bad burst over a block, we observe that the probability $\mathbb{P}\{E_i \geq k_i\}$ can be approximated with the continuous counterpart $\mathbb{P}\{B_i \geq \frac{k_i}{S_i}\}$, see Fig. 3. This approximation is reasonable when the inter-packet interval is much shorter than the typical bad burst, $\frac{1}{S} \ll \mathbb{E}\{B\}$. The continuous approximation simplifies the mathematical analysis of section IV.

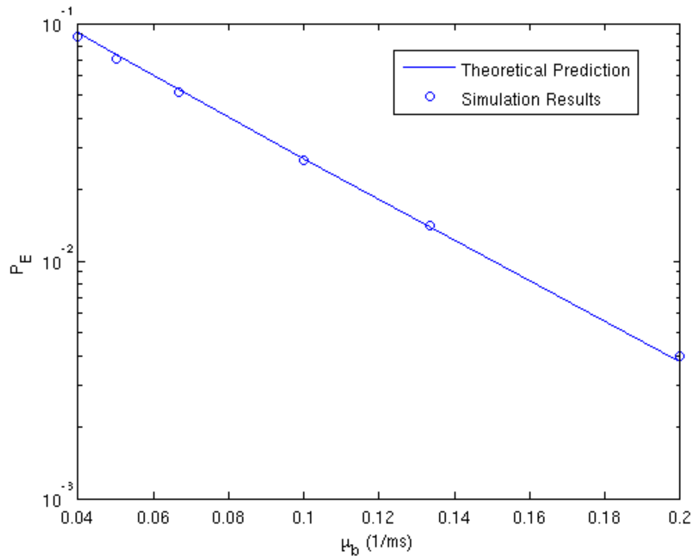


Fig. 4. Probability of irrecoverable loss versus μ_b for one path with fixed μ_g, T and α.

IV. Performance Analysis of FEC on Multiple Paths

Assume that a rate allocation algorithm assigns N_i packets to the path i. When the N_i packets of the FEC block are sent over path i, the loss count can be approximated as $\frac{B_i}{T}N_i$. Hence, The total ratio of lost packets is equal to

$$\sum_{i=1}^{L} \frac{B_i N_i}{TN} = \sum_{i=1}^{L} \frac{B_i \rho_i}{T}$$

where $\rho_i = \frac{S_i}{S_{req}},\ 0 \le \rho_i \le 1$, denotes the portion of bandwidth assigned to the path i. $x_i = \frac{B_i}{T}$ is defined as the portion of time the path i has been in the bad state ($0 \le x_i \le 1$). Hence, the probability of irrecoverable loss is equal to

$$P_E = \mathbb{P}\left\{\sum_{i=1}^{L} \rho_i x_i > \alpha\right\} \tag{8}$$

where $\alpha = \frac{N-K}{N}$. In order to find the optimum rate allocation, P_E has to be minimized with respect to the allocation vector (ρ_i's), subject to the following constraints

$$0 \le \rho_i \le \min\left\{1, \frac{W_i}{S_{req}}\right\}, \quad \sum_{i=1}^{L} \rho_i = 1 \tag{9}$$

where W_i is the TCP-friendly bandwidth defined in subsection II-C. Note that the distribution of x_i's are given and proportional to that of B_i's.

A. Performance of FEC on a Single Path

Probability of irrecoverable loss for one path is equal to

$$P_E = \mathbb{P}\{B > \alpha T\} = \mathbb{P}\{B > \alpha T|b\}\pi_b + \mathbb{P}\{B > \alpha T|g\}\pi_g$$

where $\mathbb{P}\{B > \alpha T|b\}$ and $\mathbb{P}\{B > \alpha T|g\}$ can be computed as

$$\begin{aligned} \mathbb{P}\{B > \alpha T|b\} &= \textstyle\int_{\alpha T}^{T} f_{B|b}(t)dt &&= e^{-\mu_b \alpha T}, \\ \mathbb{P}\{B > \alpha T|g\} &= \textstyle\int_{\alpha T}^{T} f_{B|g}(t)dt &&= \mu_g(1-\alpha)Te^{-\mu_b \alpha T} \end{aligned}$$

As a result

$$P_E = \pi_b e^{-\mu_b \alpha T}(1+\mu_b(1-\alpha)T) \approx \mu_g T(1-\alpha)e^{-\mu_b \alpha T}. \tag{10}$$

Fig. 4 shows the results of simulating a typical scenario of $N = 200$, $K = 180$ and $T = 200ms$ which implies the rate of $S_{req} = 1000\frac{pkt}{sec}$. μ_g is fixed at $\frac{1}{1000ms}$ and μ_b varies from $\frac{1}{5ms}$ to $\frac{1}{25ms}$, similar to the values in [2], [4]. The slope of the best linear fit (in semilog scale) to the simulation points is $-19.34ms$ which is in accordance with the value of $-19.50ms$, resulted from the theoretical approximation (10).

B. Identical Paths

When the paths are identical, due to the symmetry of the problem, the uniform rate allocation ($\rho_i = \frac{1}{L}$) is obviously the optimum solution. Of course, the solution is feasible only when for all paths, we have $\frac{1}{L} \leq w_i$. Then, the probability of irrecoverable loss can be simplified as

$$P_E = \mathbb{P}\left\{\frac{1}{L}\sum_{i=1}^{L} x_i > \alpha\right\}.$$

Let us define $Q(x)$ as the probability distribution function of x. Since x is defined as $x = \frac{B}{T}$, clearly we have $Q(x) = Tf_B(xT)$. There is a well-known upperbound for the above probability in large deviation theory [25]

$$P_E \leq e^{-u(\alpha)L}$$

$$u(\alpha) = \begin{cases} 0 & \text{for } \alpha \leq \mathbb{E}\{x\} \\ \lambda\alpha - \log(\mathbb{E}\{e^{\lambda x}\}) & \text{otherwise} \end{cases} \tag{11}$$

where λ is the solution of the following non-linear equation, which will be shown to be unique by Lemma I.

$$\alpha = \frac{\mathbb{E}\{xe^{\lambda x}\}}{\mathbb{E}\{e^{\lambda x}\}}. \tag{12}$$

Since λ is unique, we can define $l(\alpha) = \lambda$. Even though an upperbound, inequality (11) is shown to be exponentially tight for large values of L [25]. More precisely

$$P_E \doteq e^{-u(\alpha)L} \tag{13}$$

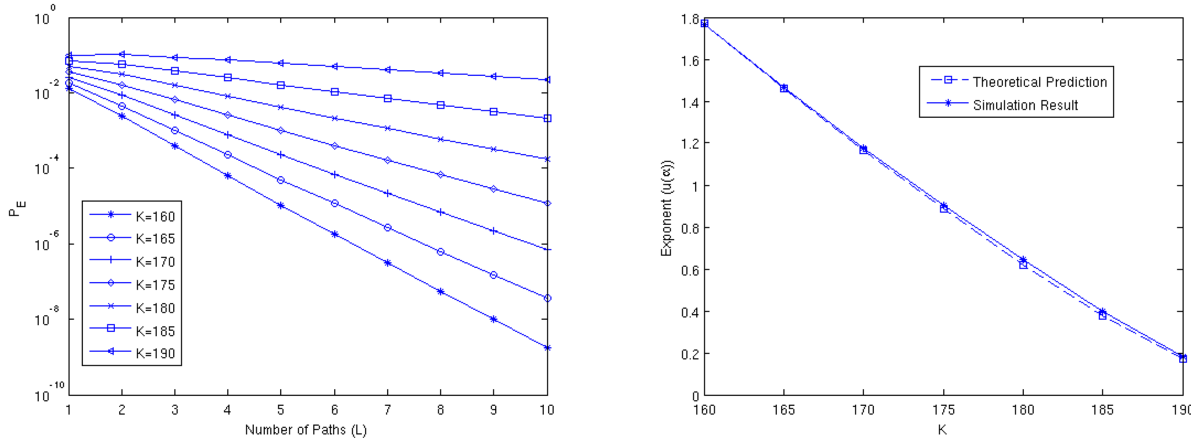
where the notation $\doteq$ means $\lim_{L->\infty} -\frac{\log P_E}{L} = u(\alpha)$. Now, we state two useful lemmas whose proofs can be found in the appendices B and C.

Lemma I. $u(\alpha)$ and $l(\alpha)$ have the following properties:

1) $\frac{\partial}{\partial\alpha}l(\alpha) > 0$
2) $l(\alpha = 0) = -\infty$
3) $l(\alpha = \mathbb{E}\{x\}) = 0$
4) $l(\alpha = 1) = +\infty$
5) $\frac{\partial}{\partial\alpha}u(\alpha) = l(\alpha) > 0$ for $\alpha > \{x\}$

Lemma II. Defining $y = \frac{1}{L}\sum_{i=1}^{L} x_i$, where x_i's are i.i.d. random variables as already defined, the probability density function of y has the property of $f_y(\alpha) \doteq e^{-u(\alpha)L}$, for all $\alpha > \mathbb{E}\{x\}$.

Fig. 5 compares the theoretical and simulation results. The connection has the aggregated bandwidth of $S_{req} = 1000pkt/s$, average bad and good burst lengths of $\frac{1}{\mu_b} = 15.0ms$ and $\frac{1}{\mu_g} = 1000ms$, and the block length of $N = 200$ packets. The number of the information packets varies from $K = 160$ to $K = 190$. The probability of irrecoverable loss is plotted versus the number of paths L in semilogarithmic scale in Fig. 5(a) for every fixed value of K. We observe that as L increases, P_E



(a) (b)

Fig. 5. (a) P_E vs. L for different values of K. (b) The Exponent (slope) of plot (a) for different values of K: experimental versus theoretical values.

decays linearly which is expectable from equation (11). Also, Fig. 5(b) compares the slope of each plot in Fig. 5(a) with $u(\alpha)$. Fig. 5 shows a good agreement between the theory and the simulation results, and also verifies the fact that the stronger the FEC code is (smaller K), the more gain we achieve from path diversity (larger exponent).

Remark I. Equation (13) is a direct result of the discrete to continuous approximation in subsection III-B. Therefore, it remains valid even if the other approximations in section III do not hold. For example, if the block time contains more than one bad burst, equations (2) and (5) are no longer valid. However, the result (13) is still true as long as the discrete to continuous approximation of is used. Of course, in this case, the real distributions of B and x should be used to compute $u(\alpha)$ and λ instead of the simplified versions.

Remark II. This analysis is valid as long as the block size is much larger than the number of paths, i.e. $N \gg L$. A special case is when the block code uses all the bandwidth resources of the paths. In this case, we have $N = LWT$, where W is the maximum bandwidth of each path, and T is the block time duration. Assuming $\alpha > \mathbb{E}\{x\}$ is a constant independent of L, we observe that $K = (1-\alpha)\,WTL$, and $P_E \doteq e^{-u(\alpha)L}$. This shows using MDS codes over multiple independent paths gives us the benefit of exponential decay of the irrecoverable loss probability and linearly growing end-to-end rate simultaneously.

C. Non-Identical Paths

Now, let us assume there are J types of paths between the source and the destination, consisting of L_j identical paths from type j ($\sum_{j=1}^{J} L_j = L$). Without loss of generality, we can assume that the paths are ordered according to their associated type, i.e. the paths from $1 + \sum_{k=1}^{j-1} L_k$ to $\sum_{k=1}^{j} L_k$ are of type j. We denote $\gamma_j = \frac{L_j}{L}$. According to the i.i.d. assumption, it is obvious that ρ_i has to be the same for all paths from the same type. η_j and y_j are defined as

$$\begin{aligned} \eta_j &= \sum_{\sum_{k=1}^{j-1} L_k < i \leq \sum_{k=1}^{j} L_k} \rho_i \\ y_j &= \frac{\eta_j}{L\gamma_j} \sum_{\sum_{k=1}^{j-1} L_k < i \leq \sum_{k=1}^{j} L_k} x_i. \end{aligned} \tag{14}$$

Following Lemma II, we observe that $f_{y_j}(\beta_j) \doteq e^{-\gamma_j u_j(\frac{\beta_j}{\eta_j})L}$. We define the sets $\mathcal{S}_I$, $\mathcal{S}_O$ and $\mathcal{S}_T$ as

$$\begin{aligned}\mathcal{S}_I &= \left\{(\beta_1, \beta_2, \cdots, \beta_J)\,|0 \le \beta_j \le 1,\ \sum_{j=1}^{J} \beta_j > \alpha\right\}\\ \mathcal{S}_O &= \left\{(\beta_1, \beta_2, \cdots, \beta_J)\,|0 \le \beta_j \le 1,\ \sum_{j=1}^{J} \beta_j = \alpha\right\}\\ \mathcal{S}_T &= \left\{(\beta_1, \beta_2, \cdots, \beta_J)\,|\eta_j \mathbb{E}\left\{x_j\right\} \le \beta_j, \sum_{j=1}^{J} \beta_j = \alpha\right\},\end{aligned}$$

respectively. Hence, P_E can be written as

$$\begin{aligned}P_E &= \mathbb{P}\left\{\sum_{j=1}^{J} y_j > \alpha\right\}\\ &= \oint_{\mathcal{S}_I} \prod_{j=1}^{J} f_{y_j}(\beta_j) d\beta_j\\ &\doteq \oint_{\mathcal{S}_I} e^{-L \sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j}{\eta_j})} d\beta_j\\ &\overset{(a)}{\doteq} e^{-L \min_{\beta \in \mathcal{S}_I \cup \mathcal{S}_O} \sum_{j=1}^{J} \gamma_j u_j\left(\frac{\beta_j}{\eta_j}\right)}\\ &\overset{(b)}{\doteq} e^{-L \min_{\beta \in \mathcal{S}_O} \sum_{j=1}^{J} \gamma_j u_j\left(\frac{\beta_j}{\eta_j}\right)}\\ &\overset{(c)}{\doteq} e^{-L \min_{\beta \in \mathcal{S}_T} \sum_{j=1}^{J} \gamma_j u_j\left(\frac{\beta_j}{\eta_j}\right)}\\ &\overset{(d)}{\doteq} e^{-L \sum_{j=1}^{J} \gamma_j u_j\left(\frac{\beta_j^\star}{\eta_j}\right)}\end{aligned} \tag{15}$$

where (a) follows from Lemma III, (b) is resulted from the fact $u_j(\alpha)$ is a strictly increasing function of α, for $\alpha > \mathbb{E}\{x_j\}$, and (c) follows from the property that $u_j(\alpha) = 0, \forall \alpha \le \mathbb{E}\{x_j\}$. Finally, applying Lemma IV results in (d).

Lemma III. For any continuous positive function $h(\mathbf{x})$ over a convex set $\mathcal{S}$, and defining $H(L)$ as

$$H(L) = \oint_{\mathcal{S}} e^{-h(\mathbf{x})L} d\mathbf{x}$$

we have

$$\lim_{L \to \infty} -\frac{\log(H(L))}{L} = \inf_{\mathcal{S}} h(\mathbf{x}) = \min_{cl(\mathcal{S})} h(\mathbf{x})$$

where $cl(\mathcal{S})$ denotes the closure of $\mathcal{S}$.

Lemma IV. There exists a unique vector $\beta^\star$ with the elements $\beta_j^\star = \eta_j l_j^{-1}\left(\frac{\nu \eta_j}{\gamma_j}\right)$ which minimizes the convex function $\sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j}{\eta_j})$ over the convex set $\mathcal{S}_T$, where ν satisfies the following condition

$$\sum_{j=1}^{J} \eta_j l_j^{-1}\left(\frac{\nu \eta_j}{\gamma_j}\right) = \alpha. \tag{16}$$

$l^{-1}()$ denotes the inverse of the function $l()$ defined in subsection IV-B. The proofs of Lemmas III and IV can be found in the appendices D and E.

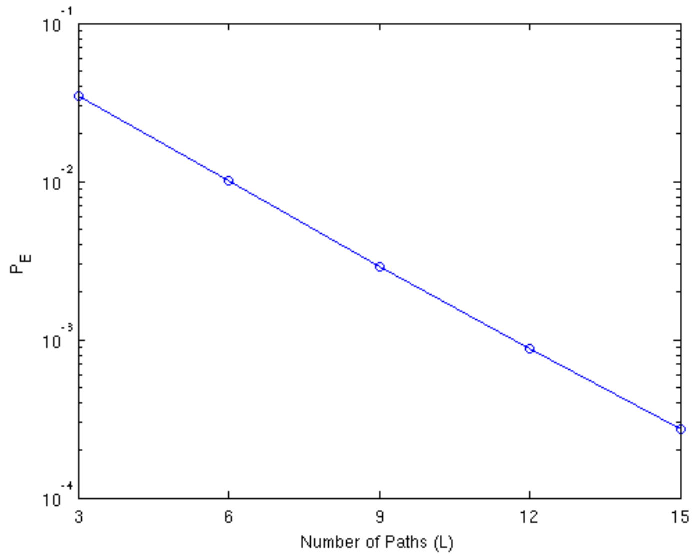


Fig. 6. P_E versus L for the combination of two path types, one third from type I and the rest from type II.

Equation (15) is valid for any fixed value of η. To achieve the most rapid decay of P_E, the exponent must be maximized over η.

$$\lim_{L->\infty} -\frac{\log P_E}{L} = \max_{0\leq\eta_j\leq 1} \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j^\star}{\eta_j}\right) \tag{17}$$

where $\beta^\star$ is defined for any value of the vector η in Lemma IV. Lemma V solves the maximization problem of (17) and identifies the asymptotically optimum rate allocation (for large number of paths).

Lemma V. The optimization problem

$$\max_{\eta} g(\eta) = \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j^\star}{\eta_j}\right),$$

$$\text{s.t.} \ \sum_{j=1}^{J} \eta_j = 1, \ 0 \leq \eta_j \leq 1$$

where $\beta^\star$ is a function of η defined in Lemma IV, has a unique solution $\eta^\star$ with the elements

$$\eta_j^\star = \begin{cases} 0 & \text{if } \alpha \leq \mathbb{E}\{x_j\} \\ \\ \dfrac{\gamma_j l_j(\alpha)}{\displaystyle\sum_{i=1,\ \alpha\leq\mathbb{E}\{x_i\}}^{J} \gamma_i l_i(\alpha)} & \text{otherwise} \end{cases} \tag{18}$$

if there is at least one $1 \leq j \leq J$ for which $\alpha > \mathbb{E}\{x_j\}$. The maximum value of the objective function is $g(\eta^\star) = \sum_{j=1}^{J} \gamma_j u_j(\alpha)$.

Fig. 6 shows P_E of the optimum rate allocation versus L for a system consisting of two types of paths. The optimal rate allocation is found by exhaustive search among all possible allocation vectors. The total aggregated rate is $S_{req} = 1000pkt/s$ and the average good burst is $\frac{1}{\mu_g} = 1000ms$ for both types. $\gamma_1 = \frac{1}{3}$ of paths benefit from shorter bad bursts with average $\frac{1}{\mu_b} = 15ms$, and the rest suffer from longer congestion bursts of $\frac{1}{\mu_b} = 25ms$. Block length is $N = 200$ for $K = 180$ information packets. The figure depicts a linear behavior in semi-logarithmic scale with the exponent of 0.403, which is comparable to 0.389 resulted from solving the maximization (17) according to Lemma V.

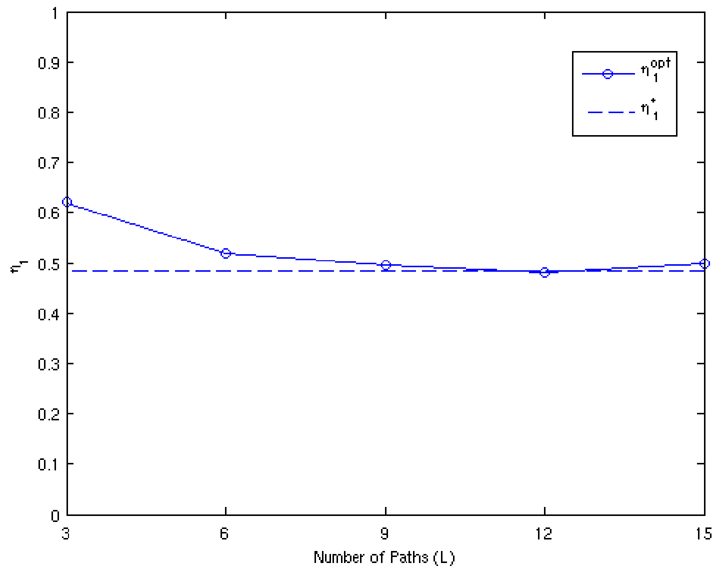


Fig. 7. The normalized aggregated weight of type I paths in the optimal rate allocation (η_1^{opt}), compared with the value of η_1 which maximizes the exponent of equation (17) ($\eta_1^{\star}$).

In the scenario of Fig. 6, let us denote the value of η_1 which maximizes equation (17) as $\eta_1^{\star}$. Obviously, $\eta_1^{\star}$ does not depend on L. Moreover, η_1^{opt} is defined as the normalized aggregated weight of type I paths in the optimal rate allocation. Fig. 7 compares η_1^{opt} with $\eta_1^{\star}$ for different number of paths. It is observed that despite some fluctuations, η_1^{opt} mainly remains independent of L when L is large enough. Fig. 6 also verifies that maximization of equation (17) actually finds the optimal allocation in the asymptotic case (L large).

V. Suboptimal Rate Allocation

In order to compute the complexity of the rate allocation problem, we go back to the original discrete formulation in subsection II-C. According to the model of subsection IV-C, we assume the available paths are from J types, L_j paths from type j, such that $\sum_{j=1}^{J} L_j = L$. Obviously, all the paths from the same type should take equal rate. Therefore, the rate allocation problem is turned into finding the vector $\mathbf{N} = (N_1, \ldots, N_J)$ such that $\sum_{j=1}^{J} N_j = N$, and $0 \leq N_j \leq L_j W_j T$ for all j. N_j denotes the number of packets to the paths of type j all together. Let us temporarily assume all paths have enough bandwidth such that N_j can vary from 0 to N for all j. There are $\binom{N+J-1}{J-1}$ L-dimensional non-negative vectors of the form $(N_1, \ldots, N_J)$ which satisfy the equation $\sum_{j=1}^{J} N_j = N$ each representing a distinct rate allocation to the different available types. Hence, the number of candidates is exponential in terms of J or N.

First, we prove the problem of rate allocation is NP in the sense that P_E can be computed in polynomial time for any candidate vector $\mathbf{N} = (N_1, \ldots, N_J)$. Let us define $P_e^{\mathbf{N}}(k, j)$ as the probability of having more than k errors over the paths of types 1 to j for a specific allocation vector $\mathbf{N}$. We also define $Q_j(n, k)$ as the probability of having exactly k errors out of the n packets sent over the paths of type j. $Q_j(n, k)$ can be computed and stored for all path types and values of n and k

with polynomial complexity as explained in appendices G and H. Then, the following recursive formula holds for $P_e^{\mathbf{N}}(k,j)$

$$\begin{aligned} P_e^{\mathbf{N}}(k,j) &= \begin{cases} \sum_{i=0}^{N_j} Q_j(N_j,i) P_e^{\mathbf{N}}(k-i,j-1) & \text{if } k \geq 0 \\ 1 & \text{if } k < 0 \end{cases} \\ P_e^{\mathbf{N}}(k,1) &= \sum_{i=k+1}^{N_1} Q_1(N_1,i). \end{aligned} \tag{19}$$

Using dynamic programming, it is obvious that the above formula computes $P_e^{\mathbf{N}}(K,J)$ with the complexity of $O(K^2 J)$.

Now, we propose a suboptimal polynomial time algorithm to find the best path allocation vector, $\mathbf{N}^{opt}$. Let us define $P_e^{opt}(n,k,j)$ as the probability of having more than k errors for a block of length n over the paths of types 1 to j minimized over all possible rate allocations ($\mathbf{N} = \mathbf{N}^{opt}$). First, we find a lowerbound $\hat{P}_e(n,k,j)$ for $P_e^{opt}(n,k,j)$ from the following recursive formula

$$\begin{aligned} \hat{P}_e(n,k,j) &= \begin{cases} \min_{0 \leq n_j \leq \min\{n, \lfloor L_j W_j T \rfloor\}} \sum_{i=0}^{n_j} Q_j(n_j,i) \hat{P}_e(n-n_j,k-i,j-1) & \text{if } k \geq 0 \\ 1 & \text{if } k < 0 \end{cases} \\ \hat{P}_e(n,k,1) &= \sum_{i=k+1}^{n} Q_1(n,i). \end{aligned} \tag{20}$$

Using dynamic programming, it is easy to show that the above formula computes $\hat{P}_e(N,K,J)$ with the complexity of $O(N^2K^2J)$. The following lemma guarantees that $\hat{P}_e(n,k,j)$ is in fact a lowerbound for $P_e^{opt}(n,k,j)$.

Lemma VI. $P_e^{opt}(n,k,j) \geq \hat{P}_e(n,k,j)$. The proof has come in the appendix I.

The following algorithm recursively finds a suboptimum allocation vector $\hat{\mathbf{N}}$ based on the lowerbound of Lemma VI.

(1): Initialize $j \leftarrow J$, $n \leftarrow N$, $k \leftarrow K$.

(2): Set

$$\begin{aligned} \hat{N}_j &= \underset{0 \leq n_j \leq \min\{n, \lfloor L_j W_j T \rfloor\}}{\operatorname{argmin}} \sum_{i=0}^{n_j} Q_j(n_j,i) \hat{P}_e(n-n_j,k-i,j-1) \\ K_j &= \underset{0 \leq i \leq \hat{N}_j}{\operatorname{argmax}} \hat{P}_e(n-\hat{N}_j,k-i,j-1) Q_j(\hat{N}_j,i) \end{aligned}$$

(3): Update $n \leftarrow n - \hat{N}_j$, $k \leftarrow k - K_j$, $j \leftarrow j-1$.

(4): If $j > 1$ and $k \geq 0$, goto (2).

(5): For $m = 1$ to j, set $\hat{N}_m \leftarrow \lfloor \frac{n}{j} \rfloor$.

(6): $\hat{N}_j \leftarrow \hat{N}_j + \operatorname{Rem}(n,j)$ where $\operatorname{Rem}(a,b)$ denotes the remainder of dividing a by b.

The following lemma guarantees that the output of the above algorithm converges to the asymptotically optimal rate allocation introduced in Lemma V of section IV-C.

Lemma VII. Consider a point-to-point connection over the Internet with L independent paths from the source to the destination, each modeled as a Gilbert-Elliot cell. The paths are from J different types, L_j paths from the type j. Assume a block FEC (N,K) is sent during an interval time T. For fixed values of $\gamma_j = \frac{L_j}{L}$, $n_0 = \frac{N}{L}$, $k_0 = \frac{K}{L}$, T and asymptotically large number of paths (L) we have

1) $\hat{P}_e(N,K,J) \doteq e^{-L \sum_{j=1}^{J} \gamma_j u_j(\alpha)}$

2) $$\frac{\hat{N}_j}{N} = \begin{cases} 0 & \text{if } \alpha \le \mathbb{E}\{x_j\} \\ \dfrac{\gamma_j l_j(\alpha)}{\displaystyle\sum_{i=1,\ \alpha \le \mathbb{E}\{x_i\}}^{J} \gamma_i l_i(\alpha)} & \text{otherwise} \end{cases}$$

3) $K_j = \alpha \hat{N}_j$ for $\alpha > \mathbb{E}\{x_j\}$.

where $\alpha = \dfrac{k_0}{n_0}$, and $u_j()$ and $l_j()$ are defined in subsections IV-B and IV-C. The proof can be found in the appendix J.

The proposed algorithm is compared with four other allocation schemes over $L = 6$ paths in Fig. 8. The optimal method uses exhaustive search over all possible allocations. '*Best Path Allocation*' assigns everything to the best path only, ignoring the rest. '*Equal Distribution*' scheme distributes the packets among all paths equally. Finally, the '*Asymptotically Optimal*' allocation assigns the rates based on the equation (18). The block length and the number of information packets are assumed to be $N = 100$ and $K = 90$, respectively. The overall rate is $S_{req} = 1000pkt/sec$, and the average good state duration is $\frac{1}{\mu_g} = 1s$ for all paths. The average duration of bad bursts for the six paths is listed as $[17.5ms \pm \frac{\Delta}{2}, 17.5ms \pm \frac{3\Delta}{2}, 17.5ms \pm \frac{5\Delta}{2}]$, such that the median is fixed at $17.5ms$. Δ is also defined as a measure of deviation from this median. $\Delta = 0$ represents the case where all paths are identical. The larger Δ is, the more variety we have among the paths and the more diversity gain might be achieved using a judicious rate allocation.

As seen in a wide range, our suboptimal algorithm tracks the optimal algorithm so closely that their graphs are not easily distinguishable. However, the '*Asymptotically Optimal*' rate allocation results in lower performance since there is only one path from each type which makes the asymptotic analysis assumptions invalid. When $\Delta = 0$, '*Equal Distribution*' scheme obviously coincides the optimal allocation. This scheme eventually diverges from the optimal algorithm as Δ grows. However, it still outperforms the best path allocation method as long as Δ is not too large. For very large values of Δ, the best path dominates all the other ones, and we can ignore the rest of the paths. Hence, the best path allocation eventually converges to the optimal scheme when Δ increases.

VI. Conclusion

In this work, we study the performance of forward error correction over a block of packets sent through multiple independent paths. First, it is shown that *Maximum Distance Separable* (MDS) block codes are optimum over our *Internet Channel* model, and any other erasure channel with or without memory, in the sense that their probability of error is minimum among all block codes of the same size. Then, probability of irrecoverable loss, P_E, is analyzed for the cases of a single path, multiple identical, and non-identical paths with a continuous approximation. When there are L identical paths, P_E is upperbounded using large deviation theory. This bound is shown to be exponentially tight in terms of L. The asymptotic analysis shows that the exponential decay of P_E with L is still valid in the case of non-identical paths. Furthermore, the optimal rate allocation problem is solved in the asymptotic case where L is very large. It is seen that in the optimal rate allocation, each path is assigned a positive rate *iff* its quality is above a certain threshold. The quality of a path is defined as the percentage of the time it spends in the bad state. In other words, including a redundant path improves the reliability *iff* this condition is satisfied. Finally, we focus on the problem of optimum path rate allocation when L is not necessarily large. A heuristic suboptimal algorithm is proposed which estimates the optimal allocation in polynomial time. For large values of L, the result of this algorithm converges to the optimal solution of the asymptotic analysis. Moreover, the simulation results verify the validity of

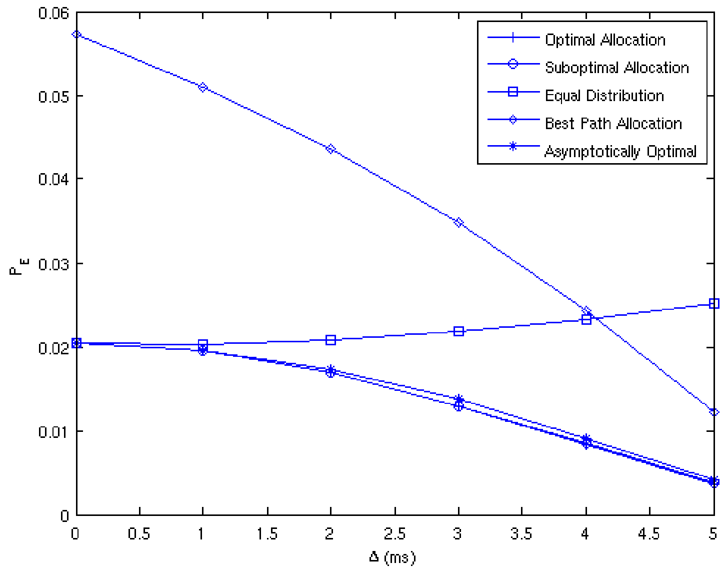


Fig. 8. Optimal and suboptimal rate allocations are compared with equal distribution and best path allocation schemes for different values of Δ

our theoretical analyses in all cases, and also show that the proposed suboptimal algorithm approximates the optimal allocation very closely.

APPENDIX A
PROOF OF PROPOSITION I

Consider a (N, K, d) codebook $\mathcal{C}$ with the q-ary codewords of length N, number of codes q^K, and minimum distance d. The distance between two codewords is defined as the number of different symbols in the same positions. A codeword $\mathbf{x} \in \mathcal{C}$ is transmitted and a vector $\mathbf{y} \in (\mathcal{X} \cup \{\xi\})^N$ is received. The binary erasure vector, $\mathbf{e}$, of length N is defined as $e_i = 1$ *iff* $y_i = \xi$, and $e_i = 0$ otherwise. Thus, the number of erased symbols is equal to the Hamming weight of $\mathbf{e}$ denoted by $w(\mathbf{e})$. Decoding error event, $\mathcal{E}$, happens when the decoder decides on a codeword different from $\mathbf{x}$. Let us assume that the probability of having a specific erasure pattern $\mathbf{e}$ is $\mathbb{P}\{\mathbf{e}\}$ which is obviously independent of the transmitted codeword, and depends only on the channel. We assume a specific erasure vector $\mathbf{e}$ of weight m happens. The decoder decodes the transmitted codeword based on the $N - m$ correctly received symbols. We partition the codebook set, $\mathcal{C}$, into q^{N-m} bins, each bin representing a specific received vector satisfying the erasure pattern $\mathbf{e}$. The number of codewords in the i'th bin is denoted by $b_{\mathbf{e}}(i)$ for $i = 1$ to q^{N-m}. Knowing the erasure vector $\mathbf{e}$ and the received vector $\mathbf{y}$, the decoder selects the bin i corresponding to $\mathbf{y}$. The set of possible transmitted codewords is equal to the set of codewords in bin i. In fact, all the codewords in bin i are equiprobable to be transmitted. If $b_{\mathbf{e}}(i) = 1$, the transmitted codeword $\mathbf{x}$ can be decoded with no ambiguity, otherwise one of the $b_{\mathbf{e}}(i) > 1$ codewords in the bin is picked randomly. Thus, the probability of error is $1 - \dfrac{1}{b_{\mathbf{e}}(i)}$ when bin i is selected. Bin i is selected if one of the codewords it contains is transmitted. Hence, probability of selecting bin i is equal to $\frac{b_{\mathbf{e}}(i)}{q^K}$. Based on the above arguments, we have

$$
\begin{aligned}
\mathbb{P}\{\mathcal{E}\} &\overset{(a)}{=} \sum_{m=d}^{N} \sum_{\mathbf{e}:w(\mathbf{e})=m} \mathbb{P}\{\mathbf{e}\}\mathbb{P}\{\mathcal{E}|\mathbf{e}\} \\
&= \sum_{m=d}^{N} \sum_{\mathbf{e}:w(\mathbf{e})=m} \mathbb{P}\{\mathbf{e}\} \sum_{i=1,\ b_{\mathbf{e}}(i)>0}^{q^{N-m}} \left(1-\frac{1}{b_{\mathbf{e}}(i)}\right)\frac{b_{\mathbf{e}}(i)}{q^K} \\
&\overset{(b)}{=} \sum_{m=d}^{N} \sum_{\mathbf{e}:w(\mathbf{e})=m} \mathbb{P}\{\mathbf{e}\} \left(1-\frac{b_{\mathbf{e}}^{+}}{q^K}\right) \\
&\geq \sum_{m=d}^{N} \sum_{\mathbf{e}:w(\mathbf{e})=m} \mathbb{P}\{\mathbf{e}\} \left(1-\frac{q^{N-m}}{q^K}\right)
\end{aligned}
\tag{21}
$$

where $b_{\mathbf{e}}^{+}$ indicates the number of bins containing one or more codewords. (a) follows from the fact that the transmitted codeword can be uniquely decoded if the number of erasures in the channel is less than the minimum distance of the codebook and (b) follows from the fact that $\sum_{i=1}^{q^{N-m}} b_{\mathbf{e}}(i) = q^K$.

According to (21), $\mathbb{P}\{\mathcal{E}\}$ is minimized for a codebook $\mathcal{C}$ *iff* two conditions are satisfied. First, the minimum distance of $\mathcal{C}$ should achieve the maximum possible value, i.e. we should have $d = N-K+1$. Secondly, we should have $b_{\mathbf{e}}^{+} = q^{N-m}$ for all possible erasure vectors $\mathbf{e}$ with any weight $d \leq m \leq N$. Any MDS code satisfies the first condition by definition. Moreover, it is easy to show that $b_{\mathbf{e}}(i) = q^{K-N+m}$ for any MDS code. We first prove this statement for the case of $m = N-K$. Consider the bins of a MDS code for any arbitrary erasure pattern $\mathbf{e}, w(\mathbf{e}) = N-K$. From the fact that $d = N-K+1$ and $\sum_{i=1}^{q^K} b_{\mathbf{e}}(i) = q^K$, it is concluded that each bin contains exactly one codeword. Therefore, there exists only one codeword which matches any K correctly received symbols. Now, consider any general erasure pattern $\mathbf{e}, w(\mathbf{e}) = m > N-K$. For the i'th bin, concatenating any $K-N+m$ arbitrary symbols to the $N-m$ correctly received symbols gives us a distinct codeword of the MDS codebook. Having q^{K-N+m} possibilities to expand the received $N-m$ symbols to K symbols, we have $b_{\mathbf{e}}(i) = q^{K-N+m}$. This completes the proof.

Appendix B

Proof of Lemma I

1) We define the function $v(\lambda)$ as

$$v(\lambda) = \frac{\mathbb{E}\{xe^{\lambda x}\}}{\mathbb{E}\{e^{\lambda x}\}}. \tag{22}$$

Then, the first derivation of $v(\lambda)$ will be

$$\frac{\partial}{\partial\lambda}v(\lambda) = \frac{\mathbb{E}\{x^2e^{\lambda x}\}\mathbb{E}\{e^{\lambda x}\} - [\mathbb{E}\{xe^{\lambda x}\}]^2}{[\mathbb{E}\{e^{\lambda x}\}]^2}. \tag{23}$$

According to Cauchy-Schwarz inequality, the following statement is always true for any two functions of $f()$ and $g()$

$$[\int_x f(x)g(x)dx]^2 < [\int_x f^2(x)dx][\int_x g^2(x)dx] \tag{24}$$

unless $f(x) = Kg(x)$ for a constant K and all values of x. If we choose $f(x) = \sqrt{x^2Q(x)e^{x\lambda}}$ and $g(x) = \sqrt{Q(x)e^{x\lambda}}$, they can not be proportional to each other for all values of x. Therefore, the enumerator of equation (23) has to be strictly positive for all λ. Since the function $v(\lambda)$ is strictly increasing, it has an inverse $v^{-1}(\alpha)$ which is also strictly increasing. Moreover, the non-linear equation $v(\lambda) = \alpha$ has a unique solution of the form $\lambda = v^{-1}(\alpha) = l(\alpha)$.

2) To show that $l(\alpha = 0) = -\infty$, we prove the equivalent statement of the form $\lim_{\lambda\to-\infty} v(\lambda) = 0$. Since x is a random variable in the range of $[0, 1]$ with the probability density function $Q(x)$, for any $0 < \epsilon < 1$ we can write

$$\begin{aligned}
\lim_{\lambda\to-\infty} v(\lambda) &= \lim_{\lambda\to-\infty} \frac{\int_0^\epsilon xQ(x)e^{x\lambda}dx + \int_\epsilon^1 xQ(x)e^{x\lambda}dx}{\int_0^1 Q(x)e^{x\lambda}dx} \\
&\le \lim_{\lambda\to-\infty} \frac{\int_0^\epsilon xQ(x)e^{x\lambda}dx}{\int_0^\epsilon Q(x)e^{x\lambda}dx} + \frac{\int_\epsilon^1 xQ(x)dx}{\int_0^\epsilon Q(x)e^{(x-\epsilon)\lambda}dx} \\
&\overset{(a)}{=} \lim_{\lambda\to-\infty} \frac{\int_0^\epsilon xQ(x)e^{x\lambda}dx}{\int_0^\epsilon Q(x)e^{x\lambda}dx} \\
&\overset{(b)}{=} \lim_{\lambda\to-\infty} \frac{x_1Q(x_1)e^{\lambda x_1}}{Q(x_2)e^{\lambda x_2}}
\end{aligned} \tag{25}$$

for some $x_1, x_2 \in [0, \epsilon]$. (a) follows from the fact that for $x \in [0, \epsilon]$, $(x - \epsilon)\lambda \to +\infty$ when $\lambda \to -\infty$, and (b) is a result of mean value theorem for integration. This theorem states that for every continuous function $f(x)$ in the interval $[a, b]$, we have

$$\exists\ x_0 \in [a, b] \quad s.t. \quad \int_a^b f(x)dx = f(x_0)[b - a]. \tag{26}$$

Equation (25) is valid for any arbitrary $0 < \epsilon < 1$. If we choose $\epsilon \to 0$, x_1 and x_2 are both squeezed in the interval $[0, \epsilon]$. Thus, we have

$$\lim_{\lambda\to-\infty} v(\lambda) \le \lim_{\lambda\to-\infty} \lim_{\epsilon\to0} \frac{x_1Q(x_1)e^{\lambda x_1}}{Q(x_2)e^{\lambda x_2}} = \lim_{\epsilon\to0} x_1 = 0 \tag{27}$$

Based on the distribution of x, $v(\lambda)$ is obviously non-negative for any λ. Hence, the inequality in (27) can be replaced by equality.

3) By observing that $v(\lambda = 0) = \mathbb{E}\{x\}$, it is obvious that $l(\alpha = \mathbb{E}\{x\}) = 0$.

4) To show that $l(\alpha = 1) = +\infty$, we prove the equivalent statement of the form $\lim_{\lambda\to+\infty} v(\lambda) = 1$. For any $0 < \epsilon < 1$ and $x \in [1 - \epsilon, 1]$, $(x - 1 + \epsilon)\lambda \to +\infty$ when $\lambda \to +\infty$. Then, defining $\zeta = 1 - \epsilon$, we have

$$\lim_{\lambda\to+\infty} \frac{\int_0^\zeta xQ(x)e^{x\lambda}dx}{\int_0^1 Q(x)e^{x\lambda}dx} \le \lim_{\lambda\to+\infty} \frac{\int_0^\zeta xQ(x)dx}{\int_\zeta^1 Q(x)e^{(x-\zeta)\lambda}dx} = 0. \tag{28}$$

Since the fraction in (28) is obviously non-negative for any λ, this inequality can be replaced by an equality. Similarly, we have

$$\lim_{\lambda\to+\infty} \frac{\int_0^\zeta Q(x)e^{x\lambda}dx}{\int_\zeta^1 xQ(x)e^{x\lambda}dx} \le \lim_{\lambda\to+\infty} \frac{\int_0^\zeta Q(x)dx}{\int_\zeta^1 xQ(x)e^{(x-\zeta)\lambda}dx} = 0. \tag{29}$$

which can also be replaced by equality. Now, the limit of $v(\lambda)$ is written as

$$\begin{aligned}
\lim_{\lambda\to+\infty} v(\lambda) &= \lim_{\lambda\to+\infty} \frac{\int_0^\zeta xQ(x)e^{x\lambda}dx + \int_\zeta^1 xQ(x)e^{x\lambda}dx}{\int_0^1 Q(x)e^{x\lambda}dx} \\
&\overset{(a)}{=} \left(\lim_{\lambda\to+\infty} \frac{\int_0^\zeta Q(x)e^{x\lambda}dx + \int_\zeta^1 Q(x)e^{x\lambda}dx}{\int_\zeta^1 xQ(x)e^{x\lambda}dx} \right)^{-1} \\
&\overset{(b)}{=} \left(\lim_{\lambda\to+\infty} \frac{\int_\zeta^1 Q(x)e^{x\lambda}dx}{\int_\zeta^1 xQ(x)e^{x\lambda}dx} \right)^{-1} \\
&\overset{(c)}{=} \left(\lim_{\lambda\to+\infty} \frac{Q(x_1)e^{x_1\lambda}}{x_2Q(x_2)e^{x_2\lambda}} \right)^{-1}
\end{aligned} \tag{30}$$

for some $x_1, x_2 \in [1 - \epsilon, 1]$. (a) and (b) follow from equations (28) and (29) respectively, and (c) is a result of mean value theorem for integration. If we choose $\epsilon \to 0$, x_1 and x_2 are both squeezed in the interval $[1 - \epsilon, 1]$. Then, equation (30) turns into

$$\lim_{\lambda\to+\infty} v(\lambda) \le \lim_{\lambda\to+\infty}\lim_{\epsilon\to 0} \frac{Q(x_1)e^{x_1\lambda}}{x_2 Q(x_2)e^{x_2\lambda}} = \lim_{\epsilon\to 0}\frac{1}{x_2} = 1.$$

5) According to equations (11) and (12), derivation of $u(\alpha)$ gives us

$$\frac{\partial u(\alpha)}{\partial\alpha} = l(\alpha) + \alpha\frac{\partial l(\alpha)}{\partial\alpha} - \frac{\mathbb{E}\{xe^{\lambda x}\}}{\mathbb{E}\{e^{\lambda x}\}}\frac{\partial l(\alpha)}{\partial\alpha} = l(\alpha).$$

Appendix C
Proof of Lemma II

Based on the definition of probability density function, we have

$$\begin{aligned}
\lim_{L\to\infty} -\frac{1}{L}\log\left(f_y(\alpha)\right) &= \lim_{L\to\infty} -\frac{1}{L}\log\left(\lim_{\delta\to\ 0}\frac{\mathbb{P}\{y>\alpha\}-\mathbb{P}\{y>\alpha+\delta\}}{\delta}\right)\\
&\ge \lim_{\delta\to\ 0}\lim_{L\to\infty}\frac{1}{L}\left[-\log\left(\mathbb{P}\{y>\alpha\}\right)+\log\delta\right]\\
&\overset{(a)}{=} u(\alpha)
\end{aligned} \tag{31}$$

where (a) follows from equation (13). The exponent of $f_y(\alpha)$ can be upper-bounded as

$$\begin{aligned}
\lim_{L\to\infty} -\frac{1}{L}\log\left(f_y(\alpha)\right) &= \lim_{\delta\to\ 0}\lim_{L\to\infty}\frac{-\log\left(\mathbb{P}\{y>\alpha\}-\mathbb{P}\{y>\alpha+\delta\}\right)+\log\delta}{L}\\
&\overset{(a)}{\le} \lim_{\delta\to\ 0}\lim_{L\to\infty}\frac{-\log\left(e^{-L(u(\alpha)+\epsilon)}-e^{-L(u(\alpha+\delta)-\epsilon)}\right)+\log\delta}{L}\\
&= \lim_{\delta\to\ 0}\lim_{L\to\infty} u(\alpha)+\epsilon-\frac{\log\left(1-e^{-L\chi}\right)}{L}\\
&\overset{(b)}{=} u(\alpha)+\epsilon
\end{aligned} \tag{32}$$

where $\chi = u(\alpha+\delta)-u(\alpha)-2\epsilon$. Since $u(\alpha)$ is an increasing function (Lemma I), we can make χ positive by choosing ϵ small enough. (a) follows from the definition of limit if L is sufficiently large, and (b) is a result of χ being positive. Making ϵ arbitrarily small, results (31) and (32) prove the lemma.

Appendix D
Proof of Lemma III

According to the definition of infimum, we have

$$\begin{aligned}
\lim_{L\to\infty} -\frac{\log(H(L))}{L} &\ge \lim_{L\to\infty} -\frac{1}{L}\log\left(e^{-L\inf\limits_{\mathcal{S}} h(\mathbf{x})}\oint_{\mathcal{S}} d\mathbf{x}\right)\\
&= \inf_{\mathcal{S}} h(\mathbf{x}).
\end{aligned} \tag{33}$$

The continuous function $h(\mathbf{x})$ must have a minimum in $cl(\mathcal{S})$ which is denoted by $\mathbf{x}^\star$. Due to the continuity of $h(\mathbf{x})$ at $\mathbf{x}^\star$, for any $\epsilon>0$, there is a neighborhood $\mathcal{B}(\epsilon)$ centered at $\mathbf{x}^\star$ such that any $\mathbf{x}\in\mathcal{B}(\epsilon)$ has the property of $|h(\mathbf{x})-h(\mathbf{x}^\star)|<\epsilon$. Moreover, we have $\mathcal{B}(\epsilon)\cap\mathcal{S}\neq\emptyset$ where $\emptyset$ denotes the empty set. Now we can write

$$\begin{aligned}
\lim_{L\to\infty} -\frac{\log(H(L))}{L} &\le \lim_{L\to\infty} -\frac{1}{L}\log\left(\oint_{\mathcal{S}\cap\mathcal{B}(\epsilon)} e^{-Lh(\mathbf{x})}d\mathbf{x}\right)\\
&\le \lim_{L\to\infty} -\frac{1}{L}\log\left(e^{-L(h(\mathbf{x}^\star)+\epsilon)}\oint_{\mathcal{S}\cap\mathcal{B}(\epsilon)} d\mathbf{x}\right)\\
&= h(\mathbf{x}^\star)+\epsilon.
\end{aligned} \tag{34}$$

Making ϵ arbitrarily small, results (33) and (34) prove the lemma.

APPENDIX E
PROOF OF LEMMA IV

According to Lemma I, $u_j(x)$ is increasing and convex for $\forall 1 \le j \le J$. Thus, the objective function $f(\beta) = \sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j}{\eta_j})$ is also convex, and the region $\mathcal{S}_T$ is determined by J convex inequality constraints and one affine equality constraint. Hence, in this case, KKT conditions are both necessary and sufficient for solution [26]. In other words, if there exist constants ϕ_j and ν such that

$$\frac{\gamma_j}{\eta_j} l_j(\frac{\beta_j^\star}{\eta_j}) - \phi_j - \nu = 0 \quad \forall 1 \le j \le J \tag{35}$$

$$\phi_j \left[\eta \mathbb{E}\{x_j\} - \beta_j^\star\right] = 0 \quad \forall 1 \le j \le J \tag{36}$$

then the point $\beta^\star$ is a global minimum.

Now, we prove that either $\beta_j^\star = \eta_j \mathbb{E}\{x_j\}$ for all $1 \le j \le J$, or $\beta_j^\star > \eta_j \mathbb{E}\{x_j\}$ for all $1 \le j \le J$. Let us assume the opposite is true, and there are at least two elements of the vector $\beta^\star$, indexed with k and m, which have the values of $\beta_k^\star = \eta_k \mathbb{E}\{x_k\}$ and $\beta_m^\star > \eta_m \mathbb{E}\{x_m\}$ respectively. For any arbitrary $\epsilon > 0$, the vector $\beta^{\star\star}$ can be defined as below

$$\beta_j^{\star\star} = \begin{cases} \beta_j^\star + \epsilon & \text{if } j = k \\ \beta_j^\star - \epsilon & \text{if } j = m \\ \beta_j^\star & \text{otherwise.} \end{cases} \tag{37}$$

Then, we have

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{f(\beta^{\star\star}) - f(\beta^\star)}{\epsilon} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \gamma_k u_k \left(\frac{\beta_k^\star + \epsilon}{\eta_k} \right) + \gamma_m u_m \left(\frac{\beta_m^\star - \epsilon}{\eta_m} \right) - \gamma_m u_m \left(\frac{\beta_m^\star}{\eta_k} \right) \right\} \\ &\overset{(a)}{=} \lim_{\epsilon \to 0} \frac{\gamma_k}{\eta_k} l_k \left(\frac{\beta_k^\star + \epsilon'}{\eta_k} \right) - \frac{\gamma_m}{\eta_m} l_m \left(\frac{\beta_m^\star + \epsilon''}{\eta_m} \right) \\ &= -\frac{\gamma_m}{\eta_m} l_m \left(\frac{\beta_m^\star}{\eta_m} \right) < 0 \end{aligned} \tag{38}$$

where $\epsilon', \epsilon'' \in [0, \epsilon]$, and (a) follows from Taylor's theorem. Thus, moving from $\beta^\star$ to $\beta^{\star\star}$ decreases the function which contradicts the assumption of $\beta^\star$ being the global minimum.

Out of the remaining possibilities, the case where $\beta_j^\star = \eta_j \mathbb{E}\{x_j\}$ ($\forall 1 \le j \le J$) obviously agrees with Lemma IV for the special case of $\nu = 0$. Therefore, the lemma can be proved assuming $\beta_j^\star > \eta_j \mathbb{E}\{x_j\}$ ($\forall 1 \le j \le J$). Then, equation (36) turns into $\phi_j = 0$ ($\forall 1 \le j \le J$). By rearranging equation (35) and using the condition $\sum_{j=1}^{J} \beta_j = \alpha$, Lemma IV is proved.

APPENDIX F
PROOF OF LEMMA V

The parameter ν is obviously a function of the vector η. Differentiation of equation (16) in Lemma IV gives us

$$\frac{\partial \nu}{\partial \eta_k} = -\frac{v_k \left(\frac{\nu \eta_k}{\gamma_k} \right) + \frac{\nu \eta_k}{\gamma_k} v_k' \left(\frac{\nu \eta_k}{\gamma_k} \right)}{\sum_{j=1}^{J} \frac{\eta_j^2}{\gamma_j} v_j' \left(\frac{\nu \eta_j}{\gamma_j} \right)} \tag{39}$$

where $v_j(x) = l_j^{-1}(x)$, and $v_j'(x)$ denotes its derivative with respect to its argument.

Let us denote the allocation vector which maximizes the objective function

$$g(\eta) = \sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j^\star}{\eta_j}) = \sum_{j=1}^{J} \gamma_j u_j \left(v_j(\frac{\nu \eta_j}{\gamma_j}) \right)$$

as $\eta^\star$. $\nu^\star$ is defined as the value of ν corresponding to $\eta^\star$. It is easy to show that $\nu^\star > 0$. Let us assume the opposite is true ($\nu^\star \leq 0$). Then, according to Lemma I, we have $v_j(\frac{\nu\eta_j}{\gamma_j}) \leq \mathbb{E}\{x_j\}$ for all j which results in $g(\eta^\star) = 0$. However, it is possible to achieve a positive value of $g(\eta)$ by setting $\eta_j = 1$ for the one vector which has the property of $\mathbb{E}\{x_j\} < \alpha$, and setting $\eta_j = 0$ for the rest. Thus, $\eta^\star$ can not be the maximal point. This contradiction proves the fact that $\nu^\star > 0$.

At the first step, we prove that $\eta_j^\star > 0$ if $\mathbb{E}\{x_j\} < \alpha$. Assume the opposite is true for an index $1 \leq k \leq J$. Since $\sum_{j=1}^J \eta_j^\star = 1$, there should be at least one index m such that $\eta_m^\star > 0$. For any arbitrary $\epsilon > 0$, the vector $\eta^{\star\star}$ can be defined as below

$$\eta_j^{\star\star} = \begin{cases} \eta_j^\star + \epsilon & \text{if } j = k \\ \eta_j^\star - \epsilon & \text{if } j = m \\ \eta_j^\star & \text{otherwise.} \end{cases} \tag{40}$$

$\nu^{\star\star}$ is defined as the corresponding value of ν for the vector $\eta^{\star\star}$. Based on equation (39), we can write

$$\begin{aligned} \Delta\nu &= \nu^{\star\star} - \nu^\star \qquad (41) \\ &= \frac{v_m\left(\frac{\nu^\star\eta_m^\star}{\gamma_m}\right) + \frac{\nu^\star\eta_m^\star}{\gamma_m} v_m'\left(\frac{\nu^\star\eta_m^\star}{\gamma_m}\right) - \mathbb{E}\{x_k\}}{\sum_{j=1}^J \frac{\eta_j^{\star 2}}{\gamma_j} v_j'\left(\frac{\nu^\star\eta_j^\star}{\gamma_j}\right)}\epsilon + o(\epsilon^2). \end{aligned}$$

Then, we have

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{g(\eta^{\star\star}) - g(\eta^\star)}{\epsilon} &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{ \frac{\nu^{\star 2}\eta_k^\star}{\gamma_k} v_k'\left(\frac{\nu^\star\eta_k^\star}{\gamma_k}\right)\epsilon - \frac{\nu^{\star 2}\eta_m^\star}{\gamma_m} v_m'\left(\frac{\nu^\star\eta_m^\star}{\gamma_m}\right)\epsilon + \nu^\star\Delta\nu \sum_{j=1}^J \frac{\eta_j^{\star 2}}{\gamma_j} v_j'\left(\frac{\nu^\star\eta_j^\star}{\gamma_j}\right) + o(\epsilon^2)\right\} \\ &\overset{(a)}{=} \nu^\star\left\{ v_m\left(\frac{\nu^\star\eta_m^\star}{\gamma_m}\right) - \mathbb{E}\{x_k\}\right\} \end{aligned} \tag{42}$$

where (a) follows from (41). If the value of (42) is positive for an index m, moving in that direction increases the objective function which contradicts with the assumption of $\eta^\star$ being a maximal point . If the value of (42) is non-positive for all indexes m whose $\eta_m^\star > 0$, we can write

$$\mathbb{E}\{x_k\} \geq \sum_{m=1}^J \eta_m^\star v_m\left(\frac{\nu^\star\eta_m^\star}{\gamma_m}\right) = \alpha \tag{43}$$

which obviously contradicts the assumption of $\mathbb{E}\{x_k\} < \alpha$.

At the second step, we prove that $\eta_j^\star = 0$ if $\mathbb{E}\{x_j\} \geq \alpha$. Assume the opposite is true for an index $1 \leq r \leq J$. Since $\sum_{j=1}^J \eta_j^\star = 1$, we should have $\eta_s^\star < 1$ for all other indices s. For any arbitrary $\epsilon > 0$, the vector $\eta^{\star\star\star}$ can be defined as below

$$\eta_j^{\star\star\star} = \begin{cases} \eta_j^\star - \epsilon & \text{if } j = r \\ \eta_j^\star + \epsilon & \text{if } j = s \\ \eta_j^\star & \text{otherwise.} \end{cases} \tag{44}$$

$\nu^{\star\star\star}$ is defined as the corresponding value of ν for the vector $\eta^{\star\star\star}$. Based on equation (39), we can write

$$\begin{aligned} \Delta\nu &= \nu^{\star\star\star} - \nu^\star \\ &= \frac{v_r\left(\frac{\nu^\star\eta_r^\star}{\gamma_r}\right) + \frac{\nu^\star\eta_r^\star}{\gamma_r} v_r'\left(\frac{\nu^\star\eta_r^\star}{\gamma_r}\right) - v_s\left(\frac{\nu^\star\eta_s^\star}{\gamma_s}\right) - \frac{\nu^\star\eta_s^\star}{\gamma_s} v_s'\left(\frac{\nu^\star\eta_s^\star}{\gamma_s}\right)}{\sum_{j=1}^J \frac{\eta_j^{\star 2}}{\gamma_j} v_j'\left(\frac{\nu^\star\eta_j^\star}{\gamma_j}\right)}\epsilon + o(\epsilon^2). \end{aligned} \tag{45}$$

Then, we have

$$\lim_{\epsilon\to 0}\frac{g(\eta^{\star\star\star})-g(\eta^\star)}{\epsilon} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\frac{\nu^{\star 2}\eta_s^\star}{\gamma_s}v_s'\left(\frac{\nu^\star\eta_s^\star}{\gamma_s}\right)\epsilon-\frac{\nu^{\star 2}\eta_r^\star}{\gamma_r}v_r'\left(\frac{\nu^\star\eta_r^\star}{\gamma_r}\right)\epsilon+\ \nu^\star\Delta\nu\sum_{j=1}^{J}\frac{\eta_j^{\star 2}}{\gamma_j}v_j'\left(\frac{\nu^\star\eta_j^\star}{\gamma_j}\right)+o(\epsilon^2)\right\}$$
$$\stackrel{(a)}{=} \nu^\star\left\{v_r\left(\frac{\nu^\star\eta_r^\star}{\gamma_r}\right)-v_s\left(\frac{\nu^\star\eta_s^\star}{\gamma_s}\right)\right\} \quad (46)$$

where (a) follows from (45). If the value of (46) is positive for an index s, moving in that direction increases the objective function which contradicts with the assumption of $\eta^\star$ being a maximal point . If the value of (46) is non-positive for all indices s whose $\eta_s^\star > 0$, we can write

$$\mathbb{E}\{x_r\} < v_r\left(\frac{\nu^\star\eta_r^\star}{\gamma_r}\right) \le \sum_{s=1}^{J}\eta_s^\star v_s\left(\frac{\nu^\star\eta_s^\star}{\gamma_s}\right) = \alpha \quad (47)$$

which obviously contradicts the assumption of $\mathbb{E}\{x_r\} \ge \alpha$.

Now that the boundary points are checked, we can safely use the KKT conditions [26] for all $1 \le k \le J$ where $\mathbb{E}\{x_k\} < \alpha$ to find the maximum η.

$$\zeta = \frac{\nu^{\star 2}\eta_k^\star}{\gamma_k}v_k'\left(\frac{\nu^\star\eta_k^\star}{\gamma_k}\right)+\nu^\star\sum_{j=1}^{J}\frac{\eta_j^{\star 2}}{\gamma_j}v_j'\left(\frac{\nu^\star\eta_j^{\star 2}}{\gamma_j}\right)\frac{\partial\nu}{\partial\eta_k}|_{\nu=\nu^\star}$$
$$\stackrel{(a)}{=} -\nu^\star v_k\left(\frac{\nu^\star\eta_k^\star}{\gamma_k}\right) \quad (48)$$

where ζ is a constant independent of k, and (a) follows from (39). Using the fact that $\sum_{j=1}^{J}\eta_j = 1$ with equations (16) and (48) gives us

$$\zeta = -\alpha\nu^\star$$
$$\nu^\star = \sum_{\mathbb{E}\{x_j\}<\alpha}\gamma_j l_j(\alpha). \quad (49)$$

Combining equations (48) and (49) proves the lemma.

Appendix G
Discrete Analysis of One Path

$Q(n,k,l)$ is defined as the probability of having exactly k errors out of the n packets sent over the path l. Since we temporarily focus on one path, the index l can be dropped in this section. Depending on the initial state of the channel, $P_g(n,k)$ and $P_b(n,k)$ are defined as the probabilities of having k errors out of the n packets when we start the transmission in good or bad states respectively. It is easy to see

$$Q(n,k) = \pi_g P_g(n,k) + \pi_b P_b(n,k). \quad (50)$$

$P_g(n,k)$ and $P_b(n,k)$ can be computed from the following recursive equations

$$P_b(n,k) = \pi_{b|b}P_b(n-1,k-1)+\pi_{g|b}P_g(n-1,k-1)$$
$$P_g(n,k) = \pi_{b|g}P_b(n-1,k)+\pi_{g|g}P_g(n-1,k) \quad (51)$$

with the initial conditions

$$P_g(n,k) = 0 \qquad \text{for} k \le n$$
$$P_b(n,k) = 0 \qquad \text{for} k > n$$
$$P_g(n,k) = 0 \qquad \text{for} k < 0$$
$$P_b(n,k) = 0 \qquad \text{for} k \le 0 \quad (52)$$

where $\pi_{s_2|s_1}$ is the probability of the channel being is in the state $s_2 \in \{g, b\}$ provided that it has been in the state $s_1 \in \{g, b\}$ when the last packet was transmitted. $\pi_{s_2|s_1}$ has the following values for different combinations of s_1 and s_2 [1]

$$\begin{aligned}
\pi_{g|g} &= \pi_g + \pi_b \; e^{-\dfrac{\mu_g + \mu_b}{S_l}} \\
\pi_{b|g} &= 1 - \pi_{g|g} \\
\pi_{b|b} &= \pi_b + \pi_g \; e^{-\dfrac{\mu_g + \mu_b}{S_l}} \\
\pi_{g|b} &= 1 - \pi_{b|b}
\end{aligned} \tag{53}$$

where S_l denotes the transmission rate on path l.

The recursive equations (51) can be solved with the complexity of $O\left(K(N-K)\right)$ which give us $Q(n,k,l)$ according to equation (50).

Appendix H
Discrete Analysis of One Type

When there are n packets to be distributed over L_j identical paths of type j, even distribution is obviously the the optimum allocation. However, since the integer n may be indivisible by L_j, the L_j dimentional vector $\mathbf{N}$ is defined as below

$$N_l = \begin{cases} \lfloor \dfrac{n}{L_j} \rfloor + 1 & \text{for } 1 \le l \le \text{Rem}(n, L_j) \\ \\ \lfloor \dfrac{n}{L_j} \rfloor & \text{for Rem}(n, L_j) < l \le L_j \end{cases} \tag{54}$$

where $\text{Rem}(a, b)$ denotes the remainder of dividing a by b. $\mathbf{N}$ represents the closest integer vector to even distribution.

$E^{\mathbf{N}}(k,l)$ is defined as the probability of having exactly k erasures among the n packets transmitted over the identical paths 1 to l with the allocation vector $\mathbf{N}$. $E^{\mathbf{N}}(k,l)$ can be computed recursively as

$$\begin{aligned}
E^{\mathbf{N}}(k,l) &= \sum_{i=0}^{k} E^{\mathbf{N}}(k-i, l-1) Q(N_l, i, l) \qquad (55) \\
E^{\mathbf{N}}(k,1) &= Q(N_1, k, 1)
\end{aligned}$$

where $Q(n,k,l)$ is obtained from appendix G According to the definitions, it is obvious that $Q_j(n,k) = E^{\mathbf{N}}(k, L_j)$ which can be calculated with the complexity of $O(k^2 n)$ using dynamic programming.

Appendix I
Proof of Lemma VI

The lemma is proved by induction on j. let us assume this statement is true for $j = 1$ to $J-1$. Then for $j = J$, we have

$$\begin{aligned}
&\hat{P}_e(n,k,J) \\
\overset{(a)}{\le}\; & \sum_{i=0}^{N_J} Q_J(N_J^{opt}, i) \hat{P}_e(n - N_J^{opt}, k-i, J-1) \\
\overset{(b)}{\le}\; & \sum_{i=0}^{N_J} Q_J(N_J^{opt}, i) P_e^{opt}(n - N_J^{opt}, k-i, J-1) \\
\overset{(c)}{\le}\; & \sum_{i=0}^{N_J} Q_J(N_J^{opt}, i) P_e^{\mathbf{N}^{opt}}(k-i, J-1) \\
\overset{(d)}{=}\; & P_e^{\mathbf{N}^{opt}}(k, J) = P_e^{opt}(n,k,J)
\end{aligned}$$

where $\mathbf{N}^{opt}$ denotes the optimum allocation of n packets among the J types of paths. (a) follows from the recursive equation (19), and (b) is the induction assumption. (c) comes from the definition of $P_e^{opt}(n,k,l)$, and (d) is a result of equation (20).

APPENDIX J
PROOF OF LEMMA VII

It is easy to see that for every n growing proportionally to L ($n = n'L$) and $k > n\mathbb{E}\{x_j\}$, we have

$$\begin{aligned} Q_j(n,k) &= \sum_{i=k}^{n} Q_j(n,i) - \sum_{i=k}^{n} Q_j(n,i) \\ &\overset{(a)}{\doteq} e^{-\gamma_j L u_j(\frac{k-1}{n})} - e^{-\gamma_j L u_j(\frac{k}{n})} \\ &\doteq e^{-\gamma_j L u_j(\frac{k}{n})} \left[e^{\gamma_j L \delta} - 1 \right] \\ &\overset{(b)}{\doteq} e^{-\gamma_j L u_j(\frac{k}{n})} \left[e^{\gamma_j \frac{L}{n} l_j(x_0)} - 1 \right] \\ &\overset{(c)}{\doteq} e^{-\gamma_j L u_j(\frac{k}{n})} \end{aligned} \tag{56}$$

where $\delta = u_j\left(\frac{k}{n}\right) - u_j\left(\frac{k-1}{n}\right)$, and $x_0 \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$. (a) follows from equation (13). (b) is a result of mean value theorem, and (c) is valid since $l_j(x_0) > 0$, and $e^{\frac{\gamma_j}{n'} l_j(x_0)}$ remains positive and finite when $L \to \infty$.

1) We prove the first part of the lemma by induction on J. When $J = 1$, the statement is obviously correct for both cases of $\frac{K}{N} > \mathbb{E}\{x_1\}$ and $\frac{K}{N} \leq \mathbb{E}\{x_1\}$, remembering the fact that $u_1(\beta) = 0$ for $\beta \leq \mathbb{E}\{x_1\}$.

Now, Lut us assume the first part of the lemma is true for $j = 1$ to $J - 1$. We try to prove the same statemenet for J as well. The proof can be divided to two different cases, depending on whether $\frac{K}{N}$ is larger than $\mathbb{E}\{x_J\}$ or vice versa.

1.1) $\frac{K}{N} > \mathbb{E}\{x_J\}$

For any $0 < \epsilon < \gamma_J$, and for all values of $\epsilon L < n_J \leq \min\{n, \lfloor L_j W_j T \rfloor\}$, there is an $0 < i_J < n_J$ where

$$\frac{i_J}{n_J} = \frac{K - i_J}{N - n_J} = \frac{K}{N} = \alpha.$$

Therefore, for $\epsilon L < n_J \leq \min\{n, \lfloor L_j W_j T \rfloor\}$, we have

$$\begin{aligned} \sum_{i=0}^{n_J} Q_J(n_J, i)\hat{P}_e(N - n_J, K - i, J - 1) &\geq Q_J(n_J, i_J)\hat{P}_e(N - n_J, K - i_J, J - 1) \\ &\overset{(a)}{\doteq} e^{-L \sum_{j=1}^{J} \gamma_j u_j(\alpha)} \end{aligned} \tag{57}$$

where (a) follows from (56).

For $0 \leq n_J \leq \epsilon L$, the number of packets assigned to the paths of type J is less that the number of such paths. Thus, one packet is allocated to n_J pf the paths, and the rest of type J paths are not used. If $\pi_{b,J}$ is defined as the probability of a path of type J being in the bad state, we can write

$$Q_J(n_J, 0) = \pi_{b,J}^{n_J} = e^{-n_J \log\left(\frac{1}{\pi_{b,J}}\right)}. \tag{58}$$

Therefore, for $0 \leq n_J \leq \epsilon L$, we have

$$\begin{aligned}\sum_{i=0}^{n_J} Q_J(n_J, i)\hat{P}_e(N-n_J, K-i, J-1) &\geq Q_J(n_J, 0)\hat{P}_e(N-n_J, K, J-1)\\ &\doteq e^{-L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K}{N-n_J}\right) - n_J \log\left(\frac{1}{\pi_{b,J}}\right)}\\ &\geq e^{-L\epsilon\log\left(\frac{1}{\pi_{b,J}}\right) - L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K}{(n_0-\epsilon)L}\right)}\\ &\stackrel{(a)}{=} e^{-L\sum_{j=1}^{J-1}\gamma_j u_j(\alpha)} \geq e^{-L\sum_{j=1}^{J}\gamma_j u_j(\alpha)} \end{aligned} \tag{59}$$

where (a) becomes valid by limiting $\epsilon \to 0$.

Inequalities (57) and (59) result in

$$\hat{P}_e(N, K, J) \;\dot{\geq}\; e^{-L\sum_{j=1}^{J}\gamma_j u_j(\alpha)} \tag{60}$$

where $H(L)\dot{\geq}e^{-hL}$ denotes

$$\lim_{L\to\infty} -\frac{\log(H(L))}{L} \leq h.$$

Combining (60) with Lemma VI proves the first part of Lemma VII for the case when $\frac{K}{N} > \mathbb{E}\{x_J\}$.

1.2) $\frac{K}{N} \leq \mathbb{E}\{x_J\}$

For any $0 < \epsilon < \gamma_J$, $0 < \theta$, and for all values of $\epsilon L < n_J \leq \min\{n, \lfloor L_j W_j T\rfloor\}$, there is an $0 < i'_J < n_J$ where $\frac{i'_J}{n_J} = \mathbb{E}\{x_J\} + \theta$. For this value of i'_J, it is not difficult to show that

$$\frac{K - i'_J}{N - n_J} < \frac{K}{N} = \alpha. \tag{61}$$

Therefore, for $\epsilon L < n_J \leq \min\{n, \lfloor L_j W_j T\rfloor\}$, we have

$$\begin{aligned}\sum_{i=0}^{n_J} Q_J(n_J, i)\hat{P}_e(N-n_J, K-i, J-1) &\geq Q_J(n_J, i'_J)\hat{P}_e(N-n_J, K-i'_J, J-1)\\ &\stackrel{(a)}{\doteq} e^{-L\gamma_J u_J(\mathbb{E}\{x_J\}+\theta) - L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K-i'_J}{N-n_J}\right)}\\ &\stackrel{(b)}{\geq} e^{-L\gamma_J u_J(\mathbb{E}\{x_J\}+\theta) - L\sum_{j=1}^{J-1}\gamma_j u_j(\alpha)}\\ &\stackrel{(c)}{=} e^{-L\sum_{j=1}^{J-1}\gamma_j u_j(\alpha)}\\ &\stackrel{(d)}{=} e^{-L\sum_{j=1}^{J}\gamma_j u_j(\alpha)} \end{aligned} \tag{62}$$

where (a) follows from (56) and the induction assumption. (b) is based on (61), and (c) can be obtained by limiting $\theta \to 0$. (d) is obvious since $u_J(\alpha) = 0$ for $\alpha \leq \mathbb{E}\{x_J\}$.

For $0 \leq n_J \leq \epsilon L$, the analysis of section 1.1 and inequality (59) are still valid. Hence, inequalities (62) and (59) result in

$$\hat{P}_e(N,K,J) \quad \dot{\geq} \quad e^{-L\sum_{j=1}^{J}\gamma_j u_j(\alpha)} \tag{63}$$

which proves the first part of Lemma VII for the case of $\frac{K}{N} \leq \mathbb{E}\{x_J\}$ when combined with Lemma VI.

2) We prove the second and third part of the lemma by induction on j for a fixed J. Assuming the two statements are true for $m = J$ to $j+1$, we try to prove the same statemenets for j as well. The proof can be divided to two different cases, depending on whether $\frac{K}{N}$ is larger than $\mathbb{E}\{x_J\}$ or vice versa. Before we proceed further into the proofs, it is helpful to introduce two new parameters N' and K' as

$$\begin{aligned} N' &= N - \sum_{m=j+1}^{J} \hat{N}_j \\ K' &= K - \sum_{m=j+1}^{J} K_j. \end{aligned}$$

According to the above definitions and the induction assumptions, it is obvious that

$$\frac{K'}{N'} = \frac{K}{N} = \alpha.$$

2.1) $\dfrac{K}{N} > \mathbb{E}\{x_j\}$

First, by contradiction, it will be shown that for any $0 < \epsilon < \gamma_j$, we have $\hat{N}_j > \epsilon L$. Let us assume the opposite is true ($\hat{N}_j \leq \epsilon L$). Then, we can write

$$\begin{aligned} \hat{P}_e(N',K',j) &\overset{(a)}{=} \sum_{i=0}^{\hat{N}_j} \hat{P}_e(N'-\hat{N}_j, K'-i, j-1) Q_j(\hat{N}_j, i) \\ &\geq \hat{P}_e(N'-\hat{N}_j, K', j-1) Q_j(\hat{N}_j, 0) \\ &\overset{(b)}{\doteq} Q_j(\hat{N}_j,0) e^{-L\sum_{r=1}^{j-1}\gamma_r u_r\left(\frac{K'}{N'-\hat{N}_j}\right)} \\ &\overset{(c)}{\geq} e^{-L\epsilon\log\left(\frac{1}{\pi_{b,J}}\right) - L\sum_{r=1}^{j-1}\gamma_r u_r\left(\frac{K'}{N'-\epsilon L}\right)} \\ &\overset{(d)}{=} e^{-L\sum_{r=1}^{j-1}\gamma_r u_r(\alpha)} \overset{(e)}{>} e^{-L\sum_{r=1}^{j}\gamma_r u_r(\alpha)} \end{aligned} \tag{64}$$

where (a) follows from equation (20) and step (2) of our suboptimal algorithm, (b) is resulted from the first part of Lemma VII, and (c) can be justified with similar arguments to those of inequality (59). (d) is obtained by limiting $\epsilon \to 0$, and (e) is valid since $u_r(\alpha)$ is strictly positive for $\alpha > \mathbb{E}\{x_J\}$. The result (64) is obviously in contradiction with the first part of Lemma VII, proving that $\hat{N}_j > \epsilon L$.

$\hat{P}_e(N', K', j)$ can be written as

$$\begin{aligned}
\hat{P}_e(N', K', j) &= \sum_{i=0}^{\hat{N}_j} \hat{P}_e(N' - \hat{N}_j, K' - i, j-1) Q_j(\hat{N}_j, i) \\
&\overset{(a)}{\doteq} \sum_{i=0}^{\hat{N}_j} e^{-L\gamma_j u_j \left(\frac{i}{\hat{N}_j}\right) - L \sum_{r=1}^{j-1} \gamma_r u_r \left(\frac{K' - i}{N' - \hat{N}_j}\right)} && (65) \\
&\overset{(b)}{\doteq} e^{-L\gamma_j u_j \left(\frac{K_j}{\hat{N}_j}\right) - L \sum_{r=1}^{j-1} \gamma_r u_r \left(\frac{K' - K_j}{N' - \hat{N}_j}\right)} && (66) \\
&\overset{(c)}{\doteq} e^{-L \sum_{r=1}^{j} \gamma_r u_r (\alpha)} && (67)
\end{aligned}$$

where (a) and (c) follow the first part of Lemma VII and (56), and (b) is based on the definition of K_j at step (2) of our suboptimal algorithm. Since the above results hold for any set of coefficients, γ, comparing (66) and (67) proves the third part of Lemma VII

$$\frac{K_j}{\hat{N}_j} = \frac{K' - K_j}{N' - \hat{N}_j} = \frac{K'}{N'} = \frac{K}{N} = \alpha. \tag{68}$$

Moreover, differentiating the exponents of (65) with respect to i to find the dominant term of the summation results in

$$\begin{aligned}
\frac{\gamma_j}{\hat{N}_j} l_j \left(\frac{K_j}{\hat{N}_j}\right) &= \sum_{r=1, \alpha > \mathbb{E}\{x_r\}}^{j-1} \frac{\gamma_r}{N' - \hat{N}_j} l_r \left(\frac{K' - K_j}{N' - \hat{N}_j}\right) && \overset{(a)}{\Rightarrow} \\
\frac{\hat{N}_j}{N - \sum_{m=j+1}^{J} \hat{N}_m} &= \frac{\gamma_j l_j(\alpha)}{\sum_{r=1, \alpha > \mathbb{E}\{x_r\}}^{j} l_r(\alpha)} && \overset{(b)}{\Rightarrow} \\
\frac{\hat{N}_j}{N} &= \frac{\gamma_j l_j(\alpha)}{\sum_{r=1, \alpha > \mathbb{E}\{x_r\}}^{J} l_r(\alpha)}
\end{aligned}$$

where (a) follows from (68), and (b) from the induction assumption.

2.2) $\frac{K}{N} \leq \mathbb{E}\{x_j\}$

First, by contradiction, it will be shown that for any $0 < \epsilon < \gamma_j$, we have $\hat{N}_j \leq \epsilon L$. Let us assume the opposite is true ($\hat{N}_j > \epsilon L$). Then, there is an $0 < i_j < \hat{N}_j$ where $\frac{i_j}{\hat{N}_j} = \mathbb{E}\{x_j\} + \theta$ for any $\theta > 0$. Then, similar to (61), we have

$$\frac{K' - i_j}{N' - \hat{N}_j} < \frac{K'}{N'} = \alpha. \tag{69}$$

Now it is easy to write

$$
\begin{aligned}
\hat{P}_e(N',K',j) &= \sum_{i=0}^{\hat{N}_j} \hat{P}_e(N'-\hat{N}_j, K'-i, j-1) Q_j(\hat{N}_j, i) \\
&\geq \hat{P}_e(N'-\hat{N}_j, K'-i_j, j-1) Q_j(\hat{N}_j, i_j) \\
&\overset{(a)}{\doteq} e^{-L\gamma_j u_j\left(\mathbb{E}\{x_j\}+\theta\right) - L\sum_{r=1}^{j-1}\gamma_r u_r\left(\frac{K'-i_j}{N'-\hat{N}_j}\right)} \\
&\overset{(b)}{>} e^{-L\gamma_j u_j\left(\mathbb{E}\{x_j\}+\theta\right) - L\sum_{r=1}^{j-1}\gamma_r u_r\left(\alpha\right)} \\
&\overset{(c)}{=} e^{-L\sum_{r=1}^{j-1}\gamma_r u_r\left(\alpha\right)} \overset{(d)}{=} e^{-L\sum_{r=1}^{j}\gamma_r u_r\left(\alpha\right)}
\end{aligned}
\tag{70}
$$

where (a) follows from the first part of Lemma VII and (56), and (b) is based on (69). (c) can be obtained by limiting $\theta \to 0$, and (d) is obvious since $u_j(\alpha) = 0$ for $\alpha \leq \mathbb{E}\{x_j\}$. The result (70) clearly contradicts the first part of Lemma VII. Thus, we should have $\hat{N}_j \leq \epsilon L$.

Since N grows proportionally to L ($N = Ln_0$), the second part of Lemma VII can be proved by making ϵ arbitrarily small and squeezing $\frac{\hat{N}_j}{N}$ in the interval $[0, \frac{\epsilon}{n_0}]$.

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