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On the Capacity of Wireless Multicast Networks

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Abstract

A wireless multicast network with a stringent decoding delay constraint and a minimum required multicast data rate is characterized. Assuming the channel state information is available only at the receiver side, and a single antenna system, the optimal expected rate achievable by a random user in the network is derived in terms of the minimum multicast requirement in two scenarios: hard coverage constraint and soft coverage constraint. In the first case, the minimum multicast requirement is expressed by multicast outage capacity while in the second case, the expected multicast rate should satisfy the minimum requirements. Also, the optimum power allocation in an infinite layer superposition code, achieving the highest expected typical rate, is derived. For the MISO case, a suboptimal coding scheme is proposed, which is shown to be asymptotically optimal, when the number of transmit antennas grows at least logarithmically with the number of users in the network.

I. Introduction

Wireless networks have recently received a considerable attention. The widespread applications of these networks, along with the specific circumstances of wireless communication, have motivated efficient transmission strategies for each application. One of these applications is data multicasting. In a wireless multicast system, a common source

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is transmitted to N users, through a fading channel. In such networks, two issues can be studied as a measure of performance: network coverage and quality of service. In the first case, the objective is to cover all the nodes in the network at least with a basic service, regardless of their channel quality. From this point of view, all the users have basically the same opportunity to receive data. However, in the second case, the average quality of service is the main objective. Therefore, users with better channel status should receive higher data rates and consequently, better quality of service. A good example for such networks is a TV broadcasting system [16]. In this system, all the subscribers are supposed to receive a basic video signal, while users with higher qualities might get additional services like high definition TV signal. The coverage issue in such systems is generally addressed as multicast minimum requirement.

Multicasting has been recently studied as a special scenario in broadcasting, where all the users are listening to a common source. In [2], the system challenges in lossy broadcasting of a common source are studied from information theoretical point of view. For an analog Gaussian source with a bandwidth equal to the channel bandwidth, analog transmission achieves the minimum average end-to-end distortion. The scenario in which the source has a larger bandwidth is studied in [3], where different methods of digital transmission are investigated. In [4], a different approach to source broadcasting, called static broadcasting, is proposed. It is assumed that all the users receive the same amount of data from a common source but with different number of channel uses, according to their channel quality. However, the actual transmission time in this definition depends on the user with the lowest channel gain, and hence, the transmission rate might be very low for large number of users.

Since the performance of a multicast network is strongly affected by the user with the worst channel condition, we are motivated to define a more *fair* approach. We consider a wireless multicast network in a slowly fading Gaussian environment. The objective is to maximize the average performance while a multicast constraint is satisfied. Average performance is defined as the service received by a randomly chosen user (typical user) in the network, while the multicast requirement is the service received by all the users.

These two requirements in a multicast network define a tradeoff, since the first one deals with a typical user of the network while the second depends on the worst channel state in the system. We assume the transmission block is large enough to yield a reliable communication. However, averaging over time is not possible because of the delay constraint. In other words, all the symbols within a transmission block experience the same channel gain. The channel state information (CSI) of each user is assumed to be known only at the receiver end. In this case, the ergodic capacity is not defined since the channel dose not have an ergodic behavior. The outage capacity [1] is defined for such channels as the maximum rate of single layered data, decodable with a high probability. In [10], a broadcast approach for a single user channel with these assumptions is proposed which optimizes the expected decodable rate. We will apply both "outage capacity" and "expected rate" definitions to characterize our network. Outage capacity is exploited when we have a hard coverage constraint on multicast data. In this case, we want to assure that a specific amount of data is conveyed within one transmission block to all the users, with a high probability. We relax the coverage constraint by stating it in terms of *expected* delivered rate to all the users within one block. In both cases we maximize the expected typical rate.

This minimum-service based approach has been studied in [6] for a single user fading channel, assuming CSI is known at the transmitter. In that work, given a service outage constraint for a real time application, the average rate is maximized for a non real time application sent on top of it. An adaptive variable rate code is proposed and shown to be optimum in that scenario. Similarly, a minimum rate constrained capacity measure is defined for broadcast channels in [5]. It is shown that the minimum rate capacity region is the ergodic capacity region of a broadcast channel, with an effective noise determined by the minimum rate requirements.

We will investigate the proposed multicast system in both SISO and MISO case. The MISO multicast asymptotical capacity limits are examined in [8], when the CSI is available at the transmitter. It is shown that the adverse effect of large number of users could be compensated by increasing the number of transmitter antennas. We will study similar scenario in our network and explore the effect of using multiple antennas.

The rest of this paper is organized as follows: in section II, the system model is elaborated. Section III and IV are specified to characterization of multicast network when we have a single antenna at the transmitter and at each receiver. In section III, we evaluate the optimal performance of the network in terms of the achievable expected typical rate and the multicast outage capacity. In other words, this section describes the hard multicast coverage constraint scenario. Section IV corresponds to a soft multicast coverage constraint, where expected multicast rate decoded in a block should satisfy the minimum requirement. In this scenario, we will explore the achievable expected typical rate. Section V investigates the MISO case, where we derive the asymptotical capacity limits for the multicast network. Finally, section VI concludes the paper.

II. System Model

In this paper, we consider a common message broadcasting network, where a single-antenna transmitter sends a common data to N single-antenna receivers. The received signal at the jth receiver, denoted by y_j can be written as

$$y_j = s_j x + n_j, \tag{1}$$

where $\{x\}$ is the transmitted signal with the total average power constraint $E[x^2] \leq \mathscr{P}$, $\{n_j\} \sim \mathcal{CN}(0,1)$ is the Additive White Gaussian Noise (AWGN) at this receiver, and $s_j \sim \mathcal{CN}(0,1)$ is the channel coefficient from the transmitter to the jth receiver. Therefore, the channel gain $h_j = |s_j|^2$ has the following CDF:

$$F_j(h) = 1 - e^{-h},$$

and is assumed to be constant during the transmission block. The typical (average) channel of the multicast network is defined as the channel of a randomly selected user. Since all the channels are i.i.d., the typical channel gain distribution is identical to that of each channel, i.e.,

$$F_{typ}(h) = F_j(h) = F(h). \tag{2}$$

Since all the N channels are Gaussian and they receive a common signal, the multicast channel is equivalent to the worst channel in the network. Due to statistical independence of the channels, the gain of that user has the following distribution:

$$\Pr\left\{\min_i(h_i) > h\right\} = (\Pr\{h_i > h\})^N = e^{-Nh}.$$

As a result, we have:

$$F_{mul}(h) = F_{\min_i(h_i)}(h) = 1 - e^{-Nh}.$$

In this paper, we are dealing with three measures defined in our network, as follows:

- the multicast outage rate, R_ϵ, the rate decodable at the multicast channel with probability $(1-\epsilon)$,
- the expected multicast rate, $R_{mul} = E_{h_{mul}}[R(h)]$, where $h_{mul} = \min_i(h_i)$, and $R(h)$ is the decodable data rate for the channel state h,
- the expected typical rate, $R_{ave} = E_{h_{typ}}[R(h)]$.

III. Hard Coverage Constraint

In this section, we consider a scenario where the multicast data has a high priority. Hence, it should be delivered to all the users in the network with a high probability $(1-\epsilon)$, where ϵ is the outage probability of the system. In this case, any loss of the multicast data by any user is defined as a coverage outage. Given this constraint, we want to maximize the average rate received by a randomly chosen user in the network. This average rate includes the expected rate of all data received by a typical user, even if the user is in outage. However, we will show that for a small enough outage probability, the users in the outage do not contribute to the expected average rate (it is optimum not to allocate them any power). In this scenario, we deal with two channels: (i) a multicast channel for which we want to guarantee an outage rate R_ϵ, and (ii) an average channel for which the highest expected rate R_{ave} is desired.

In [10], it is shown that the expected rate for a receiver with a block fading channel, unknown at the transmitter, and a stringent delay constraint, is equivalent to a weighted sum rate of a degraded broadcast channel with infinite number of receivers, each corresponding to a realization of the channel. This weighting function is $1-F(h)$, where $F(h)$ is the cumulative distribution function of the channel gain. Regarding [18], the highest rates (and subsequently the highest sum rate) of such a network is provided through superposition coding, where the rate

$$\mathrm{d}R_h = \log\left(1+\frac{h\rho(h)\mathrm{d}h}{1+hI(h)}\right) = \int_{I(h)}^{I(h)+\rho(h)\mathrm{d}h} \frac{h\mathrm{d}p}{1+hp} \tag{3}$$

is allocated to the user with gain level h. $\rho(.)$ is the power distribution function and $I(h)=\int_h^\infty \rho(u)\mathrm{d}u$, is the interference term at the channel state h. Note that, $\mathrm{d}R_h$ is not necessarily very small since our power allocation function might have some impulses in the general case. Let us define $s(p)$ as

$$s(p) = \max\left\{h|\ I(h)\geq p\right\}.$$

It is evident that this function is a decreasing function of p. According to (3), we can write the expected rate as

$$R_{ave}^{s(.)} = \int_0^\infty (1-F(h))\mathrm{d}R_h = \int_0^{\mathscr{P}} g(p,s(p))\mathrm{d}p, \tag{4}$$

where $g(x,y)=(1-F(y))\frac{y}{1+xy}$. Differentiating this function with respect to y, we get

$$\frac{\partial}{\partial y}g(x,y) = \frac{1-F(y)-yf(y)(1+xy)}{(1+xy)^2}. \tag{5}$$

Since $g(x,y)$ is a concave function of x,

$$\arg\max(g(x,y)|_{x=p}) = I_0^{-1}(p), \tag{6}$$

where $I_0(h)=\frac{(1-F(h))-hf(h)}{h^2f(h)}$. Moreover, $g(x,y)|_{x=p}$ is increasing for $y<I_0^{-1}(p)$, and decreasing elsewhere.

Let us define $P_\epsilon^{s(.)}$ for the function $s(.)$ as

$$P_\epsilon^{s(.)} = \min\left\{p|s(p)\leq h_\epsilon\right\},$$

where $h_\epsilon = F_{mul}^{-1}(\epsilon)$. For simplicity, we assume $h_\epsilon \leq 1$.With the above definitions, our problem is translated to find

$$\max{}_{s(.)} R_{ave}^{s(.)} = \max_{s(.)} \int_0^{\mathscr{P}} g(p,s(p))\mathrm{d}p, \tag{7}$$

subject to

$$R_\epsilon^{s(.)} = \int_{P_\epsilon^{s(.)}}^{\mathscr{P}} m(p, s(p))\mathrm{d}p \geq R_\epsilon,$$

where $m(x,y) = \frac{y}{1+xy}$ and $s(.)$ is a decreasing positive function. For any chosen x, $m(x,y)$ is an increasing function of y. Hence, we can write

$$R_\epsilon \leq \int_{P_\epsilon^{s(.)}}^{\mathscr{P}} m(p, h_\epsilon)\mathrm{d}p = \log\left(\frac{1+h_\epsilon \mathscr{P}}{1+h_\epsilon P_\epsilon^{s(.)}}\right) = C(P_\epsilon^{s(.)}).$$

Since $C(p)$ is a decreasing function of p,

$$P_\epsilon^{s(.)} \leq C^{-1}(R_\epsilon). \tag{8}$$

Lemma 1 *Denoting the optimizer of the problem (7) as $s^*(.)$, we have $P_\epsilon^{s^*(.)} \leq I_0(h_\epsilon)$.*

Proof: Assume $P_\epsilon^{s^*(.)} > I_0(h_\epsilon)$. Define $s^{**}(p)$ as

$$s^{**}(p) = \begin{cases} I_0^{-1}(p) & p < I_0(h_\epsilon) \\ h_\epsilon & I_0(h_\epsilon) \leq p \leq P_\epsilon^{s^*(.)} \\ s^*(p) & p > P_\epsilon^{s^*(.)} \end{cases}.$$

We can write

$$R_\epsilon^{s^{**}(.)} = \int_{I_0(h_\epsilon)}^{P_\epsilon^{s^*(.)}} m(p, s^{**}(p))\mathrm{d}p + \int_{P_\epsilon^{s^*(.)}}^{P} m(p, s^*(p))\mathrm{d}p \geq R_\epsilon^{s^*(.)}.$$

Moreover, we have

$$\begin{aligned} R_{ave}^{s^*(.)} &= \int_0^{I_0(h_\epsilon)} g(p, s^*(p))\mathrm{d}p + \int_{I_0(h_\epsilon)}^{P_\epsilon^{s^*(.)}} g(p, s^*(p))\mathrm{d}p + \int_{P_\epsilon^{s^*(.)}}^{\mathscr{P}} g(p, s^{**}(p))\mathrm{d}p \\ &\leq \int_0^{I_0(h_\epsilon)} g(p, I_0^{-1}(p))\mathrm{d}p + \int_{I_0(h_\epsilon)}^{P_\epsilon^{s^*(.)}} g(p, h_\epsilon)\mathrm{d}p + \int_{P_\epsilon^{s^*(.)}}^{\mathscr{P}} g(p, s^{**}(p))\mathrm{d}p \\ &= R_{ave}^{s^{**}(.)}, \end{aligned}$$

where the inequality is concluded from (5), (6), and the fact that $s^*(p) > h_\epsilon$, for $p \leq P_\epsilon^{s^*(.)}$. Therefore, our assumption of $s^*(.)$ being optimal is not valid and the lemma is proved. ■

The above lemma states the fact that, applying the multicast outage constraint, more power will be allocated to the channel gains lower than the outage threshold, compared to the unconstrained scenario [10], where $I_0(.)$ is the interference term which leads to the optimal expected rate.

Lemma 2 *Given $P_\epsilon^{s(.)} = \alpha$, the optimizer of (7) is given by*

$$s_\alpha^*(p) = \eta(\lambda, p) = \begin{cases} I_0^{-1}(p) & p < \alpha \\ h_\epsilon & \alpha \leq p \leq I_\lambda(h_\epsilon) \\ I_\lambda^{-1}(p) & p > I_\lambda(h_\epsilon) \end{cases}, \tag{9}$$

where $I_\lambda(h) = \frac{(\lambda+1-F(h))-hf(h)}{h^2 f(h)}$, and

$$\lambda = \begin{cases} 0, \int_\alpha^{\mathscr{P}} m(p, \eta(0,p))\mathrm{d}p > R_\epsilon \\ \arg\left(\int_\alpha^{\mathscr{P}} m(p, \eta(\lambda,p))\mathrm{d}p = R_\epsilon\right), otherwise \end{cases}.$$

Proof: It can be concluded directly from (6) that,

$$\int_0^\alpha g(p, s_\alpha^*(p))\mathrm{d}p \leq \int_0^\alpha g(p, I_0^{-1}(p))\mathrm{d}p. \tag{10}$$

Moreover, regarding the outage constraint of our problem,

$$\int_\alpha^{\mathscr{P}} g(p, s_\alpha^*(p))\mathrm{d}p = R_{max}(R_\epsilon, \alpha),$$

where $R_{max}(R_\epsilon, \alpha) = \max_{s(p)\leq h_\epsilon, p\geq\alpha} \int_\alpha^{\mathscr{P}} g(p, s(p))\mathrm{d}p$, subject to $\int_\alpha^{\mathscr{P}} m(p, s(p))\mathrm{d}p \geq R_\epsilon$. Writing K.K.T. condition, we have

$$R_{max}(R_\epsilon, \alpha) = \max_{s(p)\leq h_\epsilon, p\geq\alpha} \int_\alpha^{\mathscr{P}} T_\lambda(p, s(p))\mathrm{d}p, \tag{11}$$

where, $T_\lambda(x, y) = (g(x, y) + \lambda m(x, y))$. λ is 0, if the outage constraint is not limiting; otherwise, it could be obtained through the outage constraint $\int_\alpha^{\mathscr{P}} m(p, s(p))\mathrm{d}p = R_\epsilon$. Differentiating the function $T_\lambda(x, y)$ with respect to y, we get

$$\frac{\partial}{\partial y} T_\lambda(x, y) = \frac{\lambda + 1 - F(y) - yf(y)(1 + xy)}{(1 + xy)^2}. \tag{12}$$

Since $T_\lambda(x, y)$ is a concave function of y,

$$\arg\max(T_\lambda(x, y)|_{x=p}) = I_\lambda^{-1}(p). \tag{13}$$

Moreover, $T_\lambda(x,y)|_{x=p}$ is increasing for $y < I_\lambda^{-1}(p)$, and decreasing elsewhere. Hence, for any function $s(p)$ such that $s(p) < h_\epsilon$ for $P > \alpha$, we can write

$$\int_\alpha^{\mathscr{P}} T_\lambda(p, s(p))\mathrm{d}p \leq \int_\alpha^{\mathscr{P}} T_\lambda(p, s(\lambda, p))\mathrm{d}p. \tag{14}$$

Therefore,

$$R_{ave}^{s_\alpha^*(.)} = \int_0^{\mathscr{P}} g(p, s_\alpha^*(p))\mathrm{d}p \leq \int_0^{\mathscr{P}} g(p, s(\lambda, p))\mathrm{d}p, \tag{15}$$

and the proof of lemma is complete. ■

Theorem 3 *The solution to the optimization problem (7) can be written as*

$$\max_{s(.)} R_{ave}^{s(.)} = \max_{0 \leq \alpha \leq \min(C^{-1}(R_\epsilon), I_0(h_\epsilon))} \int_0^{\mathscr{P}} g(p, s_\alpha^*(p))\mathrm{d}p.$$

Proof: The proof is directly concluded from Lemma 1, Lemma 2, and inequality (8). ■

Corollary 1 *The capacity region of a Rayleigh fading multicast network* (R_ϵ, R_{ave})*, is bounded by* (C_ϵ, C_{ave})*, such that*

$$C_\epsilon = \log\left(1 + \frac{h_\epsilon \beta \mathscr{P}}{1 + h_\epsilon(1-\beta)\mathscr{P}}\right),$$

where β *changes from* 0 *to* 1 *and*

$$C_{ave} = 2(E_i(\theta(\beta)) - E_i(1)) - (e^{-\theta(\beta)} - e^{-1}) + e^{-h_\epsilon} C_\epsilon,$$

where $\theta(\beta) = \frac{2}{1+\sqrt{1+4(1-\beta)\mathscr{P}}}$*, and* $E_i(x) = \int_x^\infty \frac{e^{-t}}{t}\mathrm{d}t$*, for any* $\epsilon > 0$ *such that* $h_\epsilon \leq I_0^{-1}(\mathscr{P})$*.*

Proof: Since $h_\epsilon \leq I_0^{-1}(\mathscr{P})$, $I_\lambda(h_\epsilon) > \mathscr{P}$, for any $\lambda \geq 0$. Therefore, (9) leads to the optimum power distribution function

$$\rho(h) = (\mathscr{P} - \alpha)\delta(h - h_\epsilon) + A(h),$$

where

$$A(h) = \begin{cases} \frac{2}{h^3} - \frac{1}{h^2} & I_0^{-1}(\alpha) < h < 1 \\ 0 & else \end{cases}.$$

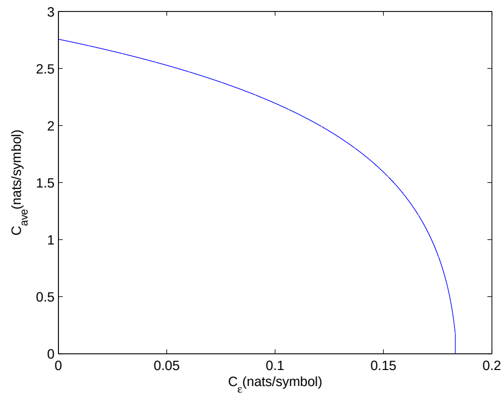

Fig. 1. Hard coverage constraint: multicast outage capacity vs. expected typical rate for $\mathscr{P} = 100$ and $N = 5$

This power distribution results in the proposed capacity region. ■

An interesting conclusion of Corollary 1 is that, the expected typical rate is maximized when the multicast rate is provided in a single layer code. In the case we have no multicast constraint, it is shown in [10] that a multilevel coding with a small rate in each level is optimal in terms of maximizing the expected rate. However, when we are constrained to distribute a fraction of power to a set of *low* channel gains $[0, h_\epsilon]$ (coverage constraint), it is optimum to allocate all the power to the highest gain (h_ϵ).

Note that the assumption $h_\epsilon \leq I_0^{-1}(\mathscr{P})$ is not hard to satisfy, since the outage probability ϵ is usually small. Moreover, the value of h_ϵ decreases significantly with the number of users, such that it could be approximated by $\frac{\epsilon}{N}$. For example, for $N = 5$ and $\mathscr{P} = 100$, the outage probability ϵ could be as high as 0.38 in order to have $h_\epsilon \leq I_0^{-1}(\mathscr{P})$. In figure (1) we can see the capacity region of this network when $\epsilon = 0.01$. It is evident that due to hard coverage constraint for all the users, the achievable outage rates are very small in comparison with the expected rate values.

IV. Soft Coverage Constraint

In the previous section, we observed that a strict coverage constraint for multicasting results in very small values of multicast rate. We can relax the coverage requirement by considering the average service received by *all* the users in *one* channel block. In fact, we can replace the *outage* requirement by the *expected* multicast rate. In this case, all the users should receive a minimum rate in average and given that, we want a typical user to receive the highest expected rate. Therefore, the measures we are dealing with in this section are R_{mul} and R_{ave}.

Lemma 4 *The highest achievable couple of* (R_{mul}, R_{ave}) *can be achieved by a superposition coding scheme.*

Proof: In [10], it is shown that the average decoded rate of a receiver with a quasi-static Gaussian fading channel, unknown at the transmitter, can always be written as a weighted sum rate of a degraded Gaussian broadcast channel. This broadcast channel has infinite number of users, each associated with a realization of the channel, where the weight of each users rate is the value of channel CDF at that realization. Now, assume a rate couple (R_1, R_2) is achieved by any scheme other than superposition coding. Since both multicast channel and average channel experience a common signal, the rates R_1 and R_2 are weighted sum of an infinite size vector $\mathbf{R}^*$ in the capacity region of the corresponding degraded broadcast channel, with weighting functions $1 - F_{ave}(h)$ and $1 - F_{mul}(h)$, respectively. It is shown in [18], that superposition coding provides higher rates than any other scheme for a degraded broadcast channel. Hence, in our broadcast channel there exists a rate vector $\mathbf{R}^+$, such that $\mathbf{R}^+ \geq \mathbf{R}^*$ and $\mathbf{R}^+$ is achieved by a superposition code. Since both weighting functions are positive, the corresponding weighted sums of $\mathbf{R}^+$, denoted by R_1^+ and R_2^+ are greater than R_1 and R_2, respectively, and the lemma is proved. ■

In fact, the above lemma implies that multicast constraint dose not affect the optimality of superposition coding to achieve the highest expected rate. As in [10], the transmitter can view an unknown channel as a continuum of receivers, experiencing different fading levels. However, in our scenario we have two of such channels. The objective is to design a continuum of code layers to provide the required expected rate in the multicast channel and maximize the expected rate in the typical channel.

Theorem 5 *The capacity region of a Rayleigh fading multicast network* (R_{mul}, R_{ave})*, is bounded by* (C_{mul},C_{ave})*, such that:*

$$C_{ave} = \int_0^\infty e^{-u} \frac{u\rho_\gamma(u)\mathrm{d}u}{1+uI_\gamma(u)}, \tag{16}$$

$$C_{mul} = \int_0^\infty e^{-Nu} \frac{u\rho_\gamma(u)\mathrm{d}u}{1+uI_\gamma(u)}, \tag{17}$$

where

$$I_\gamma(h) = \begin{cases} \mathscr{P} & \textit{if } h < h_0 \\ \frac{e^{-h}(1-h)+\gamma e^{-Nh}(1-Nh)}{h^2(e^{-h}+\gamma N e^{-Nh})} & h_0 < h < h_1 \\ 0 & h > h_1 \end{cases},$$

$\rho_\gamma(h) = -\frac{\partial I_\gamma(h)}{\partial h}$*, and* h_0 *and* h_1 *are real numbers, such that*

$$\frac{e^{-h_0}(1-h_0)+\gamma e^{-Nh_0}(1-Nh_0)}{h_0^2(e^{-h_0}+\gamma N e^{-Nh_0})} = \mathscr{P},$$

$$\frac{e^{-h_1}(1-h_1)+\gamma e^{-Nh_1}(1-Nh_1)}{h_1^2(e^{-h_1}+\gamma N e^{-Nh_1})} = 0,$$

for different positive values of γ.

Proof: According to lemma 4, in order to bound the achievable rate region of our network, we should search between different infinite layer superposition codes and find the one with the highest weighted sum rate corresponding to the weighting functions $1 - F_{ave}(h)$ and $1 - F_{mul}(h)$. Assuming $\rho(h)\mathrm{d}h$ as the power allocated to the layer associated to the channel gain h, the rate of that layer is

$$\mathrm{d}R_h = \log\left(1 + \frac{h\rho(h)\mathrm{d}h}{1+hI(h)}\right) = \frac{h\rho(h)\mathrm{d}h}{1+hI(h)}, \tag{18}$$

where

$$I(h) = \int_h^\infty \mathrm{d}u\rho(u),$$

and

$$I(0) = \mathscr{P}.$$

Using the above equation, the rate received at the receiver at the fading level h is

$$R(h) = \int_0^h \frac{u\rho(u)\mathrm{d}u}{1+uI(u)}.$$

Regarding to our definitions of multicast channel and average channel, the average rate in each of them can be written as follows:

$$R_{mul} = \int_0^\infty (1 - F_{mul}(u))\mathrm{d}R(u) = \int_0^\infty e^{-Nu} \frac{u\rho(u)\mathrm{d}u}{1 + uI(u)}, \tag{19}$$

$$R_{ave} = \int_0^\infty (1 - F_{ave}(u))\mathrm{d}R(u) = \int_0^\infty e^{-u} \frac{u\rho(u)\mathrm{d}u}{1 + uI(u)}. \tag{20}$$

Now, the problem is given $R_{mul} = r$, what is the maximum achievable R_{ave}. In other word,

$$R_{ave} = \max_{I(u)} \int_0^\infty e^{-u} \frac{u\rho(u)\mathrm{d}u}{1 + uI(u)}, \tag{21}$$

subject to:

$$\int_0^\infty e^{-Nu} \frac{u\rho(u)\mathrm{d}u}{1 + uI(u)} = r, \tag{22}$$

$$I(0) = \mathscr{P},$$

and

$$I(\infty) = 0.$$

In order to solve this optimization problem we define $S(x, I(x), I'(x), \gamma)$ as follows:

$$S(x, I(x), I'(x), \gamma) = e^{-x} \frac{x\rho(x)}{1 + xI(x)} - \gamma e^{-Nx} \frac{x\rho(x)}{1 + xI(x)}, \tag{23}$$

where

$$I'(x) = -\rho(x).$$

The necessary condition for $I(x)$ to maximize (21) with the constraint (22) is the zero functional variation [13] of $S(x, I(x), I'(x), \gamma)$,

$$\frac{\partial}{\partial I} S - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial}{\partial I'} S = 0, \tag{24}$$

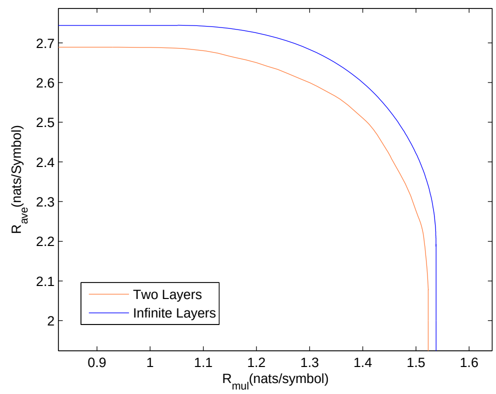

Fig. 2. Soft coverage constraint: expected multicast rate vs. expected typical rate for $\mathscr{P} = 100$ and $N = 5$

where

$$\frac{\partial}{\partial I}S = (e^{-x} - \gamma e^{-Nx})\frac{x^2 I'(x)}{(1 + xI(x))^2},$$
$$\frac{\partial}{\partial I'}S = (e^{-x} - \gamma e^{-Nx})\frac{-x}{1 + xI(x)},$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial I'}S = \frac{x(e^{-x} - \gamma N e^{-Nx})}{1 + xI(x)} + (e^{-x} - \gamma e^{-Nx})\frac{x^2 I'(x) - 1}{(1 + xI(x))^2}.$$

Therefore, (24) simplifies to a linear equation which leads to the optimum interference function given in (18).

■

Figure (2) shows the achievable rate region for $N = 5$ and $\mathscr{P} = 100$. It can be observed that the maximum average rate is achieved for multicast requirement, $R_{mul} \leq 1.05$. It is shown in [12], that a good fraction of the highest expected rate with infinite layers of code is achieved by two layers. Figure (2) shows that this is true for our multicast network as well. Furthermore, we can observe that the two-layer code region, gets closer to the capacity region at high multicast rate area. This can be justified by relative good performance of finite level codes for the channels with low variance power gain.

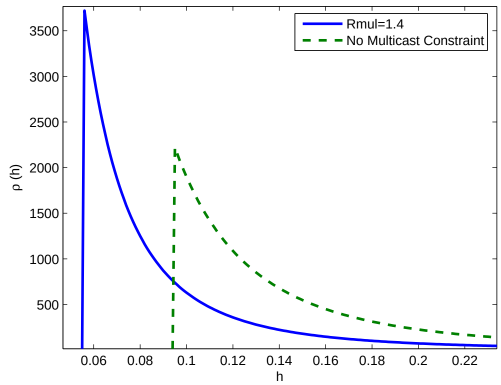

Fig. 3. Power distribution function $\rho(h)$ for no multicast requirement and for $R_{mul} = 1.4$

If we didn't have the muticast constraint and our objective was only to maximize the average rate, it is shown in [11] that the power distribution function would be,

$$\rho_{average}(h) = \begin{cases} \frac{2}{h^3} - \frac{1}{h^2} & s_0 < h < 1 \\ 0 & else \end{cases},$$

where

$$s_0 = \frac{2}{1 + \sqrt{1 + 4(P - P_1)}}.$$

This function is depicted in figure (3), and is compared with the case we have a multicast requirement $R_{mul} = 1.4$. As shown in the figure, the coverage requirement for all the users has shifted the power to lower channel gains, in order to provide service for the user with the worst channel quality in the network.

V. EXTENSION TO MISO

In the case we have multiple (M) antennas at the transmitter, we can adopt the broadcast approach proposed in [10]. In this approach, the receiver with unknown quasi-static fading MIMO channel is modeled as a continuum of receivers each associated with a channel realization. These receivers are ordered in a degraded fashion. However, since

MIMO-BC is inherently non-degraded, this approach dose not necessarily lead to the optimum performance.

Assuming single antenna at each receiver side, the ordering of the modeled receivers in this approach is based on their normalized channel norm, $\frac{||HH^{\dagger}||}{M}$. Hence, the rate of the receiver is

$$R\left(\frac{||HH^{\dagger}||}{M}\right) = \log\left(1 + \frac{P_S\frac{||HH^{\dagger}||}{M}}{1 + P_I\frac{||HH^{\dagger}||}{M}}\right) = C\left(\frac{||HH^{\dagger}||}{M}, P_I, P_S\right), \tag{25}$$

where P_S and P_I are the decodable and undecodable signal power levels, respectively.

Now, assume N users in this model, all receiving a common source through an infinite-layer code. We want to design this code to maximize the average rate received by a typical user, while providing a given rate for all the users. For this purpose, we should provide this rate for the worst user in our degraded broadcast model. This user has the lowest channel vector norm. The normalized channel norm of user i, denoted by $\frac{1}{M}||H_iH_i^{\dagger}||$, is a scaled χ_2 random variable with $2M$ degrees of freedom, whose CDF can be obtained as

$$F_{ave}(h) = F_{\frac{1}{M}||H_iH_i^{\dagger}||}(h) = 1 - \frac{\Gamma(M, Mh)}{\Gamma(M)}, \tag{26}$$

where $\Gamma(\alpha)$ is a gamma function, and $\Gamma(\alpha, \beta)$ is an upper incomplete gamma function. Since, the users' channels are statistically independent, the distribution of the norm of the weakest channel can be obtained as following:

$$\Pr\left\{\min_i \frac{1}{M}||H_iH_i^{\dagger}|| > h\right\} = \left(\Pr\left\{\frac{1}{M}||H_iH_i^{\dagger}|| > h\right\}\right)^N = \left(\frac{\Gamma(M, Mh)}{\Gamma(M)}\right)^N. \tag{27}$$

Hence, the cumulative distribution function for the weakest user's channel norm is

$$F_{mul}(h) = 1 - \left(\frac{\Gamma(M, Mh)}{\Gamma(M)}\right)^N. \tag{28}$$

Following the same approach as in section IV, the average rate and multicast rate could be written as

$$R_{mul} = \int_0^{\infty}(1 - F_{mul}(u))\mathrm{d}R(u) = \int_0^{\infty}(\frac{\Gamma(M, Mu)}{\Gamma(M)})^N \frac{u\rho(u)\mathrm{d}u}{1 + uI(u)}, \tag{29}$$

$$R_{ave} = \int_0^{\infty}(1 - F_{ave}(u))\mathrm{d}R(u) = \int_0^{\infty} \frac{\Gamma(M, Mu)}{\Gamma(M)} \frac{u\rho(u)\mathrm{d}u}{1 + uI(u)}, \tag{30}$$

where $\rho(u)$ and $I(u)$ are the corresponding power allocation and interference power functions. Defining $S(x, I(x), I'(x), \gamma)$ as

$$S(x, I(x), I'(x), \gamma) = \frac{\Gamma(M, Mx)}{\Gamma(M)} \frac{x\rho(x)}{1 + xI(x)} - \gamma \left(\frac{\Gamma(M, Mx)}{\Gamma(M)} \right)^N \frac{x\rho(x)}{1 + xI(x)}, \tag{31}$$

and setting its functional variation equal to zero to maximize the average rate, similar to (24), we obtain the optimizer $I_\gamma(x)$ as

$$I_\gamma(h) = \begin{cases} \mathscr{P} & \text{if } h < h_0 \\ \frac{\Gamma(M,Mh) - \gamma \frac{\Gamma(M,Mh)^N}{\Gamma(M)^{N-1}}}{Mh^{M+1} e^{-Mh} (1 - \gamma N \frac{\Gamma(M,Mh)}{\Gamma(M)}^{N-1})} - \frac{1}{h} & h_0 < h < h_1 \\ 0 & h > h_1 \end{cases}, \tag{32}$$

where h_0, h_1 and γ are obtained through the following equations, respectively:

$$I_\gamma(h_0) = \frac{\Gamma(M, Mh_0) - \gamma \frac{\Gamma(M,Mh_0)^N}{\Gamma(M)^{N-1}}}{Mh_0^{M+1} e^{-Mh_0} \left(1 - \gamma N (\frac{\Gamma(M,Mh_0)}{\Gamma(M)})^{N-1} \right)} - \frac{1}{h_0} = \mathscr{P}, \tag{33}$$

$$I_\gamma(h_1) = \frac{\Gamma(M, Mh_1) - \gamma \frac{\Gamma(M,Mh_1)^N}{\Gamma(M)^{N-1}}}{Mh_1^{M+1} e^{-Mh_1} \left(1 - \gamma N (\frac{\Gamma(M,Mh_1)}{\Gamma(M)})^{N-1} \right)} - \frac{1}{h_1} = 0, \tag{34}$$

$$R_{multicast} = \int_0^\infty \left(\frac{\Gamma(M, Mu)}{\Gamma(M)} \right)^N \frac{u\rho_\gamma(u) \mathrm{d}u}{1 + uI_\gamma(u)} = r. \tag{35}$$

The achievable rate region is shown in figure (4) for different values of N, when $M = 2$. As mentioned in [10], the idea of modeling the unknown fading channel by a degraded broadcast channel with infinite number of receivers is not when we have multiple antennas. This is mainly because a MIMO broadcast channel is not degraded. One may claim that the same model with a general MIMO broadcast channel and the capacity region proposed in [17] might outperform our model. However, since we have a common message broadcasting, all the data decoded at a transmitter is important for us, even the part treated as the interference in the Broadcast Channel. In other words, we are utilizing the degraded characteristic of the channel as we are assuming it receives whatever a weaker receiver decodes, plus its corresponding data. As a result, there would be some limitation for applying a general MIMO Broadcast model.

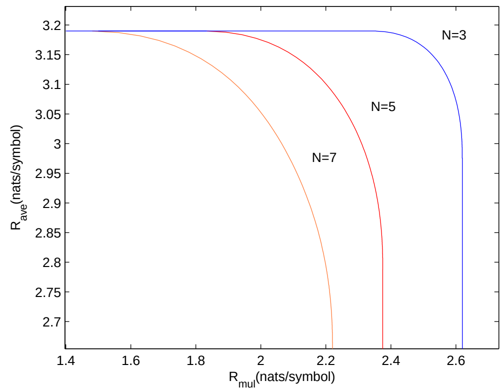

Fig. 4. Soft coverage constraint: MISO expected multicast rate vs. expected typical rate for different number of users, $M = 2$ and $\mathscr{P} = 100$

In figure (4), we can see that as the number of users decreases, the proposed achievable rate region expands more. It is also evident by comparing the region of MISO and SISO (figure(2)) channels with $N = 5$ users, that using multiple antennas improves the achievable rates. However, its effect on the achievable rates for the multicast channel is more considerable than for the average channel. This prominent gain for multicast channel is sensible, since we are using multiple independent paths to convey the data, so the probability of having very low channel gains for all paths (which mainly corresponds to multicast channel) significantly decreases. In fact, we will show that we can compensate the adverse effect of number of users by increasing the number of transmit antennas. More specifically, if both N and M tend to infinity and M grows highly enough with respect to N, we will show that the multicast rate could reach the average rate and our scheme gives the optimal solution, although it is not for small number of transmit antennas. The following theorem states this fact.

Theorem 6 *For large values of M and N, the proposed infinite layer superposition coding will provide R_{mul}, such that*

$$R_{mul} \geq R_{opt} - \delta, \tag{36}$$

if

$$M > \frac{\mathscr{P}^2 \log(N) + \omega(1)}{(1+\mathscr{P})^2 \delta^2}, \tag{37}$$

where R_{opt} is the highest achievable average rate for a randomly selected user in the network and δ is an arbitrarily small positive number.

Proof: First of all, we propose an upper bound for the achievable average rate for a randomly selected user, by assuming no stringent delay constraint, meaning that the transmission block can be chosen as long as the fading block. In this case, the channel has an ergodic behavior, so that ergodic capacity is defined and is shown to be:

$$C_{erg} = E\left[\log\left(1 + \frac{||HH^{\dagger}||}{M}\mathscr{P}\right)\right]. \tag{38}$$

As a result,

$$R_{opt} \leq C_{erg}. \tag{39}$$

Regarding the central limit theorem [19], the distribution of $\frac{||HH^{\dagger}||}{M}$, where

$$\frac{1}{M}||HH^{\dagger}|| = \frac{{h_1}^2 + {h_2}^2 + ... + {h_M}^2}{M} \tag{40}$$

and h_i's are independent rayleigh distributions with unit variance and unit mean, approaches to a Gaussian distribution with the CDF:

$$F_{\frac{||HH^{\dagger}||}{M}}(h) = Q\left(\frac{h-1}{\frac{1}{\sqrt{M}}}\right), \tag{41}$$

and consequently the CDF of multicast channel will be

$$F_{mul}(h) = 1 - Q\left(\frac{h-1}{\frac{1}{\sqrt{M}}}\right)^N. \tag{42}$$

Using the concavity of $\log$ function, and having the fact that $E\left[\frac{||HH^{\dagger}||}{M}\right] = 1$, we have

$$C_{erg} \leq \log(1+\mathscr{P}). \tag{43}$$

We will show that our scheme provides a multicast rate arbitrarily close to this upper bound, if we use enough number of transmit antennas. Since this upper bound is larger than the average rate the theorem will be proved. For this purpose, we use a single-layer coding. We know that our scheme outperforms this scheme, as the single-layer coding is a special case of superposition coding. Using a single-layer code with power $\mathscr{P}$ and rate R_δ, where

$$R_\delta = \log(1 + \mathscr{P}(1 - \delta')), \tag{44}$$

and

$$\delta' = \frac{(1 + \mathscr{P})\delta}{\mathscr{P}},$$

the average multicast rate in our network will be

$$R_{mul} = Pr\left\{\frac{||HH^\dagger||_{mul}}{M} > 1 - \epsilon'\right\} R_\delta, \tag{45}$$

where $||HH^\dagger||_{mul} = \min_i ||H_i H_i^\dagger||$. Regarding (42), the above equation can be written as

$$R_{mul} = Q(-\sqrt{M}\epsilon')^N R_\delta = \left[1 - Q(\sqrt{M}\delta')\right]^N R_\delta. \tag{46}$$

Assuming M large enough to have $\sqrt{M}\delta' >> 1$, and consequently $Q(\sqrt{N}\delta') << 1$, we can rewrite the above equation as

$$R_{mul} = e^{-NQ(\sqrt{M}\delta')} R_\delta. \tag{47}$$

Now, using the approximation

$$Q(x) \approx \frac{1}{\sqrt{2\pi x}} e^{-\frac{x^2}{2}} \tag{48}$$

for large values of x, we can write

$$Q(\sqrt{M}\delta') \leq e^{-M\delta'^2}. \tag{49}$$

Therefore, having

$$M \sim \frac{\log(N) + \omega(1)}{\delta'^2}, \tag{50}$$

incurs

$$NQ(\sqrt{M}\delta') \sim o(1), \tag{51}$$

and as a result,

$$\lim_{N\to\infty} R_{mul} - R_\delta = 0. \tag{52}$$

Moreover, assuming $\delta \ll 1$, (44) can be written as,

$$\begin{aligned} R_\delta &\simeq \log(1+\mathscr{P}) - \frac{\mathscr{P}\delta'}{1+\mathscr{P}} \\ &\geq C_{erg} - \delta, \end{aligned} \tag{53}$$

where the second line results from (43). Combining (39), (52), and (53), the result of Theorem 6 easily follows. ■

VI. Conclusion

We have considered a multicast channel, where a common data is transmitted from a sender to several users. It is assumed that a minimum service must be provided for all the users. For this setup, we have optimized the average service received by a typical user in the network. Two scenarios are considered for the coverage constraint. In the case of hard coverage constraint, the minimum multicast requirement is stated in terms of an outage rate received by all the users in a single transmission block. For small enough outage probabilities, it is shown that the optimal rate region is achieved by providing the required multicast rate in a single layer code, and designing an infinite-layer code as in [10], on top of it. In the case of soft coverage constraint, the multicast requirement is expressed in terms of the expected multicast rate received by all the users. An infinite layer superposition coding is shown to achieve the capacity region (C_{mul}, C_{ave}). We have also proposed a suboptimal coding scheme for the MISO multicast channel. This scheme is shown to be asymptotically optimal, when the number of transmit antennas grows at least logarithmically with the number of users.

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