PAPR Reduction Using Integer Structures in OFDM Systems

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Technical Report UW-E&CE#2004-06

April 30, 2004

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Abstract

In this work, the problem of reducing the Peak to Average Power Ratio (PAPR) in an Orthogonal Frequency Division Multiplexing (OFDM) system is considered. We design a cubic constellation, called the Hadamard constellation, whose boundary is along the bases defined by the Hadamard matrix in the transform domain. Then, we further reduce the PAPR by applying the Selective Mapping technique. The encoding algorithm is derived from a decomposition, known as the Smith Normal Form (SNF), and has a minimal complexity. This new technique offers a PAPR that is approximately 2dB to 3dB lower than that of the best known techniques without any lose in terms of energy and/or spectral efficiency and without any side information being transmitted. Moreover, it has a low computational complexity.

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) is a multicarrier transmission technique which is widely adopted in different communication applications. OFDM prevents Inter Symbol Interference (ISI) by inserting a guard interval and mitigates the frequency selectivity of a multi-path channel by using a simple equalizer. This simplifies the design of the receiver and leads to inexpensive hardware implementations. Moreover, OFDM offers some advantages in higher order modulations and in the networking operation that position OFDM as the technique of choice for the next generation of wireless networks. However, OFDM systems have the undesirable feature of a large Peak to Average Power Ratio (PAPR) of the transmitted signals. Consequently to prevent the spectral growth of the OFDM signal, the transmit amplifier must operate in its linear regions. Therefore, power amplifiers with a large linear range are required for OFDM systems, but such amplifiers will continue to be a major cost component of OFDM systems. Consequently, reducing the PAPR is pivotal to reducing the expense of OFDM systems.

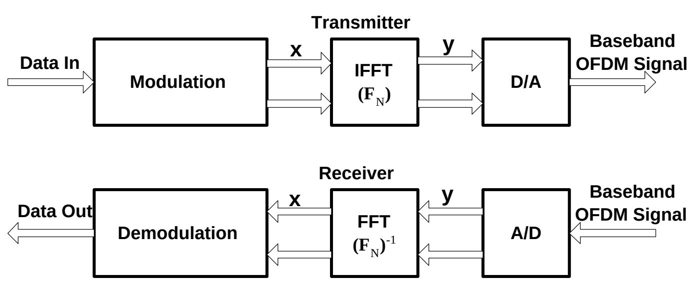

Fig. 1. Basic OFDM transmitter and receiver.

Fig. 1 shows a basic block diagram of an OFDM transmitter and receiver. Let $\mathbf{x} = \{x_0, x_1, \cdots, x_{N-1}\}$ denote a vector of $2N$ Dimensional ($2N$-D) constellation point selected from a set of N identical 2-D sub-constellations, $\{x_0, x_1, \cdots, x_{N-1}\}$, to be transmitted by using one OFDM vector of size N; namely, $\mathbf{y}$.

The discrete time samples of the OFDM signal can be expressed as

$$y_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{j2\pi \frac{nk}{N}}. \tag{1}$$

The matrix representation of this signal is

$$\mathbf{y} = \mathbf{F}_N \mathbf{x}, \tag{2}$$

where $\mathbf{y} = (y_0 \cdots y_{N-1})^T$, $\mathbf{x} = (x_0 \cdots x_{N-1})^T$, and $\mathbf{F}_N$ is the **IFFT** matrix,

$$\mathbf{F}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & e^{j2\pi \frac{nk}{N}} & \cdots & e^{j2\pi \frac{n(N-1)}{N}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & e^{j2\pi \frac{k(N-1)}{N}} & \cdots & e^{j2\pi \frac{(N-1)^2}{N}} \end{bmatrix}. \tag{3}$$

The 2-D constellation points, $\{x_0, x_1, \cdots, x_{N-1}\}$, may add constructively and produce a time domain signal with a large amplitude. Thus, the output signal $\mathbf{y}$ can have high output levels, which leads to the requirement of an expensive analog front end.

Usually, the level of the amplitude fluctuation of the discrete time OFDM signal is measured in terms of the peak factors that indicate the ratio of the peak power to the average envelop power of the signal as

$$\text{PAPR}(\mathbf{y}) = \frac{\max_i |y_i|^2}{E_y\left[\frac{1}{N}\|\mathbf{y}\|^2\right]}. \tag{4}$$

Also, the continuous time PAPR is typically estimated by the discrete time PAPR by employing the IFFT of length LN for the zero padded sequence of length LN derived from the sequence $\{x_0, x_1, \cdots, x_{N-1}\}$ in (1) [1–3]. Therefore,

$$y_n = y(nT) = \frac{\sqrt{L}}{\sqrt{LN}} \sum_{k=0}^{LN-1} x'_k e^{j2\pi \frac{nk}{LN}}, \tag{5}$$

where

$$x'_k = \begin{cases} x_k, & \text{for } k < N, \\ 0, & \text{for } k \geq N. \end{cases} \tag{6}$$

and L is the oversampling factor.

Lastly, (7) is the real representation of the complex equation in (2).

$$\begin{bmatrix} \mathrm{Re}(\mathbf{y}) \\ \mathrm{Im}(\mathbf{y}) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(\mathbf{F}_N) & -\mathrm{Im}(\mathbf{F}_N) \\ \mathrm{Im}(\mathbf{F}_N) & \mathrm{Re}(\mathbf{F}_N) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\mathbf{x}) \\ \mathrm{Im}(\mathbf{x}) \end{bmatrix}. \tag{7}$$

Since we focus on only real matrices, we use the same notation as (2) for (7).

A large number of methods for the PAPR reduction have been proposed [3–18]. In [4,5], coding techniques are used for PAPR reduction; however, codes offering a low PAPR can be constructed only at the cost of sacrificing the data rate.

Clipping the OFDM signal before amplification is a simple and typical method for the PAPR reduction [6–8]. In [3, 9, 10], the effect of oversampled and clipped OFDM signals are analyzed.

There are two probabilistic schemes to reduce the PAPR. One is the Partial Transmit Sequence (PTS) [11] in which each block of subcarriers is multiplied by a constant phase factor, and these phase factors are optimized to minimize the PAPR. The other scheme is Selective Mapping (SLM) in which multiple sequences are generated from the same information, and the sequence with the lowest PAPR is transmitted [12–14]. Typically, the receiver needs to know which sequence is selected in order to recover the data. However, the methods introduced in [11–14] eliminate the need for this explicit side information.

In [15], a new constellation technique is developed. It extends the outer constellation points to minimize the PAPR of the OFDM symbol. In [16–18], a constellation shaping technique is proposed to reduce the PAPR of the OFDM signals. The encoding and decoding algorithms of this method are based on the relations and generators in a free Abelian group. Due to the large complexity of this algorithm, its practical implementation is very challenging.

In this paper, we proposed a constellation as a shaping method in an OFDM system with a low complexity encoding method, based on [16], and a considerable PAPR reduction. An SLM method is applied in conjunction with our constellation to further reduce the PAPR in the OFDM signals.

The rest of this paper is organized as follows. In Section II, constellation shaping is introduced. A brief description of the work in [16] is also given. Section III describes the Hadamard constellation as a shaping method in OFDM systems. Some issues of the encoding and decoding algorithms are also investigated. An SLM method is applied to the Hadamard constellation in Section IV. Section V is devoted to some numerical results and a comparison of our method with some recent works. The paper is concluded in Section VI.

II. CONSTELLATION SHAPING

In constellation shaping, a constellation in the frequency domain must be found such that the resulting shaping region in the time domain has a low PAPR. A new constellation shaping method is introduced in [16–18] by Kwok and Jones. Based on the encoding algorithm introduced in [16], we propose a cubic constellation, along with an SLM method to reduce the PAPR in an OFDM system.

In a PAPR reduction problem, the peak value of the signal vector should be bounded by a specified value $\|\mathbf{y}\|_\infty \leq \beta$. If the real time signal is related to the real frequency constellation point by $\mathbf{y} = \mathbf{A}\mathbf{x}$, this inequality on the time domain boundary translates to a parallelotope[1] in the frequency domain, defined by $\mathbf{A}^{-1}$. The points inside this parallelotope are used as constellation points in transmitting the OFDM signals. The principal challenge in constellation shaping is to find a unique way to map the input data to the constellation points such that the mapping (encoding) and its inverse (decoding) can be implemented by a reasonable complexity. In [16], the relations and generators in a free Abelian group in the integer domain are used so that the parallelotope corners lie in the integer lattice. Therefore, the constellation boundary is based on a parallelotope, defined by $\mathbf{Q}_N = [\alpha \mathbf{A}^{-1}]$, where $[.]$ represents rounding, and α is the smallest value to have the same constellation points as those of the unshaped constellation.

[1] The parallelotope bases are defined along the columns of $\mathbf{A}^{-1}$.

The shaped constellation for an OFDM system is the quotient group, $\mathcal{Z}^N/\Lambda(\mathbf{Q}_N)$, where $\mathcal{Z}^N$ is the N-D integer space and $\Lambda(\mathbf{Q}_N)$ is the lattice defined by $\mathbf{Q}_N$. The encoding of this constellation is performed by the decomposition of the relation matrix $\mathbf{Q}_N$, based on column and row operations [16]. Indeed, in the mathematical literature this decomposition is known as the Smith Normal Form (SNF) of an integer matrix [19].

The SNF is a diagonalization of a matrix in the integer domain. Introduced by Smith [20], this concept has been used in many applications in many fields such as solving linear diophantine equations, finding the permutation equivalence and similarity of matrices, determining the canonical decomposition of the finitely generated Abelian groups, integer programming, computing additional normal forms, including Frobenius and Jordan normal forms, and separable computing of the discrete Fourier transform. For more historical remarks and applications of the SNF, see [21–23].

The SNF is well known theoretically, but can be difficult to compute in practice because of the potential for rapid growth in the size of the intermediate expressions. However, there are a number of algorithms to compute the SNF decomposition of an integer matrix, and there are some polynomial time algorithms for this decomposition in special cases [22, 24–27]. The following theorem and its corresponding algorithms can be used, instead of the column and row operations in [16].

Theorem 1: Any integer matrix $\mathbf{Q}_N$ can be decomposed into $\mathbf{Q}_N = \mathbf{UDV}$, where $\mathbf{D}$ is diagonal with the entries $\{\sigma_i\}_{i=1}^N$ such that $\sigma_1 \mid \sigma_2 \mid \cdots \mid \sigma_N$, and $\mathbf{U}$ and $\mathbf{V}$ are unimodular matrices. The matrix $\mathbf{D}$ is called the SNF of the matrix $\mathbf{Q}_N$ [19].
The condition $\sigma_1 \mid \sigma_2 \mid \cdots \mid \sigma_N$ in Theorem 1 is defined for finding a unique decomposition and can be ignored in the encoding procedure.

The complexity of this algorithm is the result of the computation of the SNF decomposition for the matrix $\mathbf{Q}_N$ which is based on rounding off the scaled version of the IFFT matrix. We can use the SNF decomposition methods for the encoding procedure; however, the computational complexity for OFDM systems that are defined by the IFFT matrix remains very high.

In [17], it is shown that if the matrix $\mathbf{Q_N}$ is replaced by the Hadamard matrix, the encoding and decoding algorithm for the constellation based on this matrix can be implemented by a butterfly structure that uses only bit shifting and logical AND. This simplicity is hidden in the following recursive formula for the Hadamard matrix:

$$\mathbf{H}_{2^n} = \begin{bmatrix} \mathbf{H}_{2^{n-1}} & \mathbf{H}_{2^{n-1}} \\ \mathbf{H}_{2^{n-1}} & -\mathbf{H}_{2^{n-1}} \end{bmatrix}, \text{ where } \mathbf{H}_1 = [1]. \tag{8}$$

The SNF of (8) can be easily computed in (9).

$$\begin{gathered} \mathbf{U}_{2^n} = \begin{bmatrix} \mathbf{U}_{2^{n-1}} & 0 \\ \mathbf{U}_{2^{n-1}} & \mathbf{U}_{2^{n-1}} \end{bmatrix} \ \mathbf{D}_{2^n} = \begin{bmatrix} \mathbf{D}_{2^{n-1}} & 0 \\ 0 & 2\mathbf{D}_{2^{n-1}} \end{bmatrix} \\ \mathbf{V}_{2^n} = \begin{bmatrix} \mathbf{V}_{2^{n-1}} & \mathbf{V}_{2^{n-1}} \\ 0 & -\mathbf{V}_{2^{n-1}} \end{bmatrix} \mathbf{U}_{2^n}^{-1} = \begin{bmatrix} \mathbf{U}_{2^{n-1}} & 0 \\ -\mathbf{U}_{2^{n-1}} & \mathbf{U}_{2^{n-1}} \end{bmatrix}, \end{gathered} \tag{9}$$

where $\mathbf{U}_1 = \mathbf{U}_1^{-1} = \mathbf{D}_1 = \mathbf{V}_1 = [1]$.

Therefore, the encoding algorithm for this constellation can be represented by [16]

$$\begin{aligned} \hat{\mathbf{x}} &= \mathbf{U}_N \boldsymbol{\lambda} \\ \boldsymbol{\gamma} &= \left\lfloor \frac{\mathbf{H}_N^T \hat{\mathbf{x}}}{N} \right\rfloor \\ \mathbf{x} &= \hat{\mathbf{x}} - \mathbf{H}_N \boldsymbol{\gamma} \\ \mathbf{y} &= \text{ IFFT } (\mathbf{x}), \end{aligned} \tag{10}$$

where $N = 2^n$, and $\boldsymbol{\lambda}$ is the canonical representation of integers I representing the constellation points. The canonical representation of any integer can be calculated by the recursive modulo operations; namely,

$$\begin{aligned} \lambda_1 &= I \bmod \sigma_1 \\ I_1 &= \frac{I - \lambda_1}{\sigma_1} \\ \lambda_i &= I_{i-1} \bmod \sigma_i \\ I_i &= \frac{I_{i-1} - \lambda_i}{\sigma_i}, \end{aligned} \tag{11}$$

where $1 \leq i \leq N$.

The reverse operation for finding I from the N-D vector $\mathbf{x}$ is

$$\begin{aligned} &\boldsymbol{\lambda} = \mathbf{U}_N^{-1}\mathbf{x} = (\lambda_1, \lambda_2, \cdots, \lambda_N)^T, \\ &\tilde{\lambda}_i = \lambda_i \bmod \sigma_i\ , \\ &I = \tilde{\lambda}_1 + \sigma_1(\tilde{\lambda}_2 + \sigma_2(\cdots(\tilde{\lambda}_{N-1} + \sigma_{N-1}\tilde{\lambda}_N)\cdots)). \end{aligned} \tag{12}$$

III. Hadamard Constellation in OFDM Systems

As mentioned in Section II, if the IFFT operations in OFDM multicarrier modulation could be changed by the Hadamard matrix, a very simple encoding algorithm would result. However, this type of multicarrier modulation is not very popular because it does not offer all the advantages of conventional OFDM systems [28]. The constellation that should be used in an OFDM system has a boundary along the bases of the IFFT matrix, but the encoding of containing constellation points cannot be easily implemented. We propose a cubic constellation, called the Hadamard constellation, for an OFDM system whose boundary is along the bases defined by the Hadamard matrix in the transform domain. The IFFT and Hadamard matrices are both orthogonal matrices, and therefore, the constellations along these orthogonal bases are a rotated version of each other. This idea is illustrated in Fig. 2. By substituting the proper constellation along the IFFT matrix by the Hadamard matrix in an OFDM system, the resulting PAPR is reduced; however, the encoding of this constellation, based on the SNF decomposition of the Hadamard matrix, is simple and practical. Moreover, the encoding algorithm can be implemented by a butterfly structure that uses bit shifting and logical AND structures [16].

The advantage of using the Hadamard constellation is not only a simple encoding algorithm with a low PAPR, but also the Hadamard constellation's ability to be concatenated with other methods. This motivates us to apply a Selective Mapping (SLM) technique [29, 30] to the Hadamard constellation in an OFDM system. In typical SLM methods [29, 30], the major PAPR reduction is achieved by the first few redundant bits. Employing more redundant bits necessitates a high level of complexity to obtain modest improvements in the PAPR value. However, in the proposed SLM method, employing the Hadamard constellation causes a considerable PAPR reduction by itself. As a result, this method, by just using one or two redundant bits in SLM, significantly outperforms the other PAPR reduction techniques, reported in the literature. The details of this method will be explained in the next section and will be confirmed by simulation results. In the following, we have investigated some issues that have emerged regarding the use of the Hadamard constellation in an OFDM system.

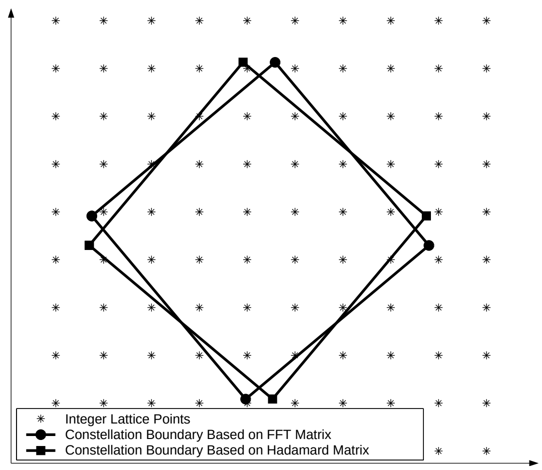

Fig. 2. N-D signal constellation for IFFT and Hadamard matrix.

A. Complex Approximation

As stated in Section II, (7) can be applied to change the complex equations of an OFDM system to real equations. This leads to the change of the constellation boundary. Generally, we can distinguish between two classes of boundaries [31, 32]: 1) Cartesian boundary that is resulted by viewing the real and imaginary parts of the signal as two separate real signals, and 2) polar boundary that considers the envelope and phase of the OFDM signal in a complex plain. Cartesian boundary limits each component of the complex signal within a square, while the polar boundary limits this component within a circle. In this paper, we avoid the complex representation of the OFDM signal by treating the real and imaginary parts of the signal independently, which is equivalent to using a Cartesian boundary.

B. Encoding Procedure

All the points inside the Hadamard constellation should be labeled by the encoding procedure, introduced in (10) - (12). The number of points, inside the shaped constellation is determined by the determinant of the Hadamard matrix, $\det(\mathbf{H}_{2^n})$.

Theorem 2: The constellation size for a $2^n \times 2^n$ Hadamard matrix is $\det(\mathbf{H}_{2^n}) = 2^{n2^{n-1}}$.

Proof: Based on (9), $\det(\mathbf{H}_{2^n}) = \det(\mathbf{D}_{2^n})$, because the matrices $\mathbf{U_{2^n}}$ and $\mathbf{V_{2^n}}$ are unimodular and their determinant is one. To prove this theorem, we use deduction. For a 2×2 Hadamard matrix,

$$\det(\mathbf{D}_2) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 2 = 2^{1\times 2^{1-1}}. \tag{13}$$

It is assumed that the claim is valid for a $2^k \times 2^k$ Hadamard matrix. Based on (9), for a $2^{k+1} \times 2^{k+1}$ Hadamard matrix,

$$\mathbf{D}_{2^{k+1}} = \begin{bmatrix} \mathbf{D}_{2^k} & 0 \\ 0 & 2\mathbf{D}_{2^k} \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow \det(\mathbf{D}_{2^{k+1}}) = \det(\mathbf{D}_{2^k}) \times 2^{2^k} \times \det(\mathbf{D}_{2^k}) \\ &\Rightarrow \det(\mathbf{D}_{2^{k+1}}) = 2^{2^k} \times (\det(\mathbf{D}_{2^k}))^2 \\ &\Rightarrow \det(\mathbf{D}_{2^{k+1}}) = 2^{2^k} \times \left(2^{k2^{k-1}}\right)^2 = 2^{(k+1)2^{(k+1)-1}}. \end{aligned} \tag{14}$$

According to the large Hadamard constellation size, the canonical representation of the large numbers should be computed in the transmission of the OFDM signals. The canonical representation can be simplified by using the fact that digital communication systems deal with binary input streams. Based on (12) in the encoding procedure, any Integer I can be represented by

$$I = \lambda_1 + \sigma_1\lambda_2 + \sigma_1\sigma_2\lambda_3 + \cdots + \sigma_1 \ldots \sigma_{N-1}\lambda_N, \tag{15}$$

where $N = 2^n$, and $\{\lambda_i\}_{i=1}^N$ is the canonical representation of I with $\lambda_1 = 0$. Based on (9), for a $2^n \times 2^n$ Hadamard matrix, all $\{\sigma_i\}_{i=1}^{2^n}$ are powers of 2 such that

$$\{\sigma_i\}_{i=1}^{2^n} = \{1, 2, 2, 4, 2, 4, 4, 8, \cdots, 2^n\}. \tag{16}$$

Therefore,

$$I = \lambda_2 + 2\lambda_3 + 2^2\lambda_4 + 2^4\lambda_5 \cdots + 2^{n2^{n-1}-n}\lambda_N. \tag{17}$$

The representation of $d = 2^{n2^{n-1}}$ integers necessitate that $n_b = \log_2(d) = n2^{n-1}$ bits, and thus, the binary representation of I is expressed as

$$\begin{aligned} I = &\ b_0 + 2b_1 + 2^2b_2 + 2^3b_3 + \cdots + 2^{n_b-1}b_{n_b-1} \\ = &\ b_0 + 2b_1 + 2^2(b_2 + 2b_3) + 2^4b_4 + 2^5(b_5 + b_6) + \\ &\cdots + 2^{n2^{n-1}-n}(b_{n_b-n} + \cdots + 2^{n-1}b_{n_b-1}) \end{aligned} \tag{18}$$

A comparison of (17) and (18) is depicted in Fig. 3. This representation will simplify the encoding algorithm. Moreover, the problem of using large numbers in the encoding procedure will be avoided.

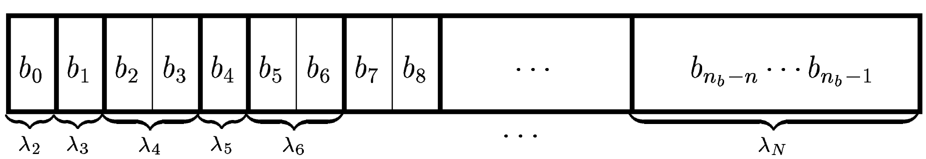

Fig. 3. Mapping between binary representation of the information and $\{\lambda_i\}$.

Theorem 2 shows the size of the Hadamard constellation for a $2^n \times 2^n$ Hadamard matrix is $2^{n2^{n-1}}$. Therefore, the transmission rate is related to the number of subcarriers $N = 2^n$ in the OFDM system[2]. This rate is unacceptable not only because of the dependency on N but also because the value is usually much higher than the required rate. A

[2]For $N = 2^n$, the rate for each real component is $\log_2(2^{n2^{n-1}})/N = \frac{n}{2}$.

selection method for the constellation points must be implemented such that the constellation has the desired rate, and the constellation points have a uniform distribution in the cubic constellation.

Noting (10) and (11), there is an isomorphism between the integer set

$$\mathcal{S} = \left\{0, 1, \cdots, 2^{n2^{n-1}} - 1\right\} \tag{19}$$

and the set of the points within the Hadamard constellation. Equivalently, the set $\mathcal{S}$ can be considered as a label group for the constellation points (refer to [33] for the definition). A subgroup of constellation points results in a uniformly distributed subset of the Hadamard constellation points. Consequently, this subgroup of constellation points is isomorphic to a subgroup in the label group $\mathcal{S}$. This subgroup can be selected such that its elements are congruent to zero modulo c, namely

$$\mathcal{P} = \{I \in \mathcal{S} \mid I \equiv 0 \pmod{c}\}, \tag{20}$$

where c is determined by the ratio of the size of the Hadamard constellation, $2^{n2^{n-1}}$, and the size of the constellation, 2^{rN}, with the desired rate, r. Employing (10) and (11), the labels in the subgroup $\mathcal{P}$ determine the set of uniformly distributed points in the Hadamard constellation. Relying on the continuous approximation, such a uniform distribution affects neither the probabilistic behavior of the PAPR nor the average energy of the constellation points. Note that the Hadamard constellation is called the root constellation for the aforementioned set of the uniformly distributed points in the sequel.

C. Decoding Procedure

A conventional Fast Fourier Transform (FFT) based receiver is considered for the OFDM signal. At the receiver end, the time domain signal is filtered by a low pass filter and sampled at the Nyquist rate. The samples are processed by an FFT to recover the constellation point in the frequency domain. For an Additive White Gaussian Noise (AWGN) channel, the received vector is given by

$$\mathbf{z} = \mathbf{y} + \mathbf{n}, \tag{21}$$

where $\mathbf{y}$ is the transmitted time domain signal in (10) and $\mathbf{n}$ is a zero-mean complex AWGN. The approximated constellation point is written as

$$\hat{\mathbf{x}} = \text{FFT}(\mathbf{z}) = \mathbf{x} + \text{FFT}(\mathbf{n}) = \mathbf{x} + \mathbf{n}', \tag{22}$$

where $\mathbf{x}$ is the transmitted constellation point, and $\mathbf{n}'$ is a zero-mean complex AWGN. The maximum likelihood decoder simply rounds off the received constellation point $\hat{\mathbf{x}}$ in the integer domain. Then, the resulting constellation point is employed in (12) to decode the transmitted information.

IV. SELECTIVE MAPPING

As mentioned in Section III, the complexity of using the Hadamard constellation in an OFDM system is very low, and this shaping method can be concatenated by other methods. In the following, we propose an SLM technique, applied to the Hadamard constellation to further reduce the PAPR.

SLM is a method to reduce the PAPR in an OFDM system which involves generating a large set of data vectors that represent the same information, where the data vector with the lowest PAPR is used for the transmission. In some SLM methods, additional information about which data is used should be transmitted to the receiver end. This will cause potential problems with decoding the signal in the presence of noise, and will obviously result in a loss in the transmission rate. We present a method to apply the SLM technique to further reduce the PAPR in the constellation developed earlier, in which the need for the transmission of side information is removed.

Assume that the data rate to be transmitted is r bits per block of length-N FFT symbol. Let r_s denote the number of redundant bits of r bits specified for SLM ($r_s \ll r$ and $r = \log_2(\det(\text{constellation size}))$). Consequently, $N_s = 2^{r_s}$ constellation points should represent the same information. In this method, the input integers I are mapped to the Hadamard constellation points, and the output integers are classified by the sets with the same r_s Most Significant Bits (MSBs). All the corresponding constellation points in each set represent the same information. The IFFT operation for all these constellation points in each set is computed, and the constellation point with the lowest PAPR is transmitted.

The operation of our scheme can be described as follows. In the first step, a binary information sequence is divided into blocks of $r - r_s$ bits. r_s bits of zeros are added to each information block, and then it is divided into subblocks of lengths equal to $\{\log_2 \sigma_i\}$ bits (refer to Fig. 3). The binary representations of these subblocks form the vector $\boldsymbol{\lambda}$ in (10). The other multiples of this vector are obtained by changing all the possible values for r_s MSBs of the binary information sequence. Then, N_s different Hadamard constellation points are produced by (10). The corresponding time domain OFDM signals result in various values for the PAPR. Finally, the constellation point with the lowest PAPR is selected for transmission.

All the different constellation points that represent the same information have the same $r - r_s$ bits. Thus, at the receiver end, the constellation point is decoded by (12), and the r_s extra bits are discarded, since the transmitted information is in the remaining $r - r_s$ bits. Therefore, this method can be expressed as a variant of SLM in which no side information on the choice of the transmit signal needs to be transmitted. The degradation in the data rate can be ignored, since by using only one or two redundant bits, which are fewer than those of data rate, a significant PAPR reduction is obtained.

The Hadamard constellation has a zero shaping gain[3], due to its cubic boundary. Numerical results show applying the SLM method to the resulting cubic constellation and selecting the point with the lowest PAPR result in a reduction in the average energy, reflected in a small, however positive shaping gain. This justifies our earlier claim that the reduction in the PAPR is achieved at no extra cost in terms of a reduction in the spectral efficiency and/or an increase in the average energy of the constellation.

[3]Shaping gain is defined as the relative reduction in the required average energy for a given number of constellation points with respect to a cubic constellation.

V. SIMULATION RESULTS

In this section, we present simulations for a complex baseband OFDM system with $N = 128$ subchannels employing 16-QAM by using 10^7 randomly generated OFDM symbols. First, we show the PAPR performance of the Hadamard constellation. The next step is then to show the capability of the SLM technique, when it is applied to the Hadamard constellation to achieve a significant PAPR reduction. Our simulation results are presented as the Complementary Cumulative Density Function (CCDF) of the PAPR of the OFDM signals. This is expressed as follows:

$$\text{CCDF}\left\{\text{PAPR}(\mathbf{y})\right\} = P\left\{\text{PAPR}(\mathbf{y}) > \gamma\right\}. \tag{23}$$

This equation can be interpreted as the probability that the PAPR of a symbol block exceeds some clip level γ (it is referred as symbol clip probability [15]).

As mentioned in Section I, the PAPR is a major problem for the time domain signal, that is, the effect of a large PAPR is in the continuous signal rather than in the discrete signal. According to (5) and (6), the continuous PAPR can be estimated by the IFFT of length LN for the zero padded sequence of length LN. Results for the oversampling to $L = 1, 2,$ and 4 are reflected in Fig. 4. The continuous PAPR can be approximated by the oversampling $L = 4$. As mentioned in [1–3], further oversampling will result in minor improvements only. We have a PAPR reduction of more than 4dB close to the 10^{-5} symbol clip probability.

Increasing the OFDM block length N causes the PAPR of the conventional OFDM signal to increase. As plotted in Fig. 5 for the different values of $N = 32, 64,$ and 128, the PAPR of the OFDM system changes very little. The effect of the constellation size, when the Hadamard constellation is used in the OFDM system is also investigated. Like conventional OFDM systems, employing different constellations in each subchannel does not affect the PAPR of the OFDM signal, since all the constellation points have the same root constellation. Employing different constellations in subchannels is equivalent to having various data rates. As mentioned before, all the constellations with the lower number of points are selected from a root constellation that is defined by the Hadamard matrix.

This result is exhibited in Fig. 6. Employing other modulation schemes in the previous simulations results in minor changes only of all the depicted PAPR statistics. According to these simulations, the use of the Hadamard constellation in OFDM systems as a constellation shaping method considerably reduces the PAPR with a low complexity encoding and decoding algorithm which can be easily implemented.

Fig. 7 offers the simulation results of implementing our SLM technique, applied to the Hadamard constellation in the simulated OFDM system. The PAPR probability for $r_s = 1, 2,$ and 4 redundant bits is depicted.
As it is illustrated in Fig. 7, using only one bit in 4×128 bits per block of length 128 FFT symbol [4] results in a 5.6dB improvement in the PAPR reduction; more redundant bits further reduces the PAPR.

A. Some Insight to the Achieved Performance

In a conventional OFDM system with N different subcarriers, the time domain samples can be approximated by zero mean Gaussian random variables, based on adopting the central limit theorem. Therefore, the amplitude of these samples has a Rayleigh distribution, and the CCDF of the PAPR of the OFDM signal can be approximated as follows:

$$P\left\{\text{PAPR}(\mathbf{y}) > \gamma\right\} = 1 - (1 - e^{-\gamma})^N. \tag{24}$$

The use of N_s statistically independent vectors that have the same information for transmission in the SLM method changes the CCDF of the PAPR of the OFDM signal such that

$$P\left\{\text{PAPR}(\mathbf{y}) > \gamma\right\} = \left(1 - (1 - e^{-\gamma})^N\right)^{N_s}. \tag{25}$$

Therefore, in the logarithmic CCDF vs. PAPR graph, the slope of the depicted line is proportional to N_s (see Fig. 8). By increasing the number of vectors with the same information, the slope of the CCDF vs. PAPR graph increases. Thus, the major PAPR reduction

[4] By using 16-QAM in a 128 channel OFDM system, there are $16^{128} = 2^{4\times 128}$ constellation points.

is gained by the first few redundant bits, and the PAPR reduction which is gained by the successively doubling of N_s is lower in each step, as it is shown in Fig. 8 ($\Delta_1 > \Delta_2 > \cdots$). This is the reason that we have applied the SLM technique to the Hadamard constellation. As mentioned in Section IV, the method employing only the Hadamard constellation considerably reduces the PAPR. By Adopting the Hadamard constellation in the proposed SLM method, not only can we lower the PAPR considerably, but also we can approximately maintain the slope of the CCDF vs. PAPR curve. Therefore, by using just one or two redundant bits, we can further reduce the PAPR. By using eight redundant bits in the general SLM methods in an OFDM system, we attain the same PAPR reduction as that we have achieved by using only one redundant bit in the proposed SLM method applied to the Hadamard constellation.

B. Comparison

To complete our simulations, we compare our results with some of those in recent works. In [12], an SLM method based on multiplying the constellation point by U different and pseudo-random but fixed vectors is introduced. For the same system as ours, with $N_s = 4$ different vectors, the PAPR reduction of 3dB close to 10^{-5} symbol clip probability is gained; however, at the same symbol clip rate, we have a 4dB PAPR reduction just by using the Hadamard constellation and 6dB by using our SLM method. Also, the complexity of our algorithm is comparable with the method in [12]. The main complexity is in the encoding procedure, but due to the special recursive structure of the Hadamard matrix, the encoding algorithm can be implemented easily. Also, note that in [12] some side information needs to be sent, and receiving accurate side information is important.

In [12], a PTS method is introduced. The PAPR reduction of this method is also less than that in our method. It is possible to apply a PTS method to our constellation, especially since the PTS is considerably better with respect to PAPR reduction vs. additional system complexity (the number of IFFTs) [11, 12].

The tone reservation, a well known method for PAPR reduction in multicarrier systems [34], is an efficient PAPR reduction technique, provided it can converge quickly to a good PAPR solution. In [32], an efficient approximation for the active-set approach is developed, and an excellent cost vs. performance tradeoff is obtained by using the octagonal boundary. The method in [32], compared to the work in [34], has almost the same performance in the PAPR reduction, and a faster convergence. The complexity of [32] is comparable with ours; however, we have about 3dB lower PAPR than that in [34] or [32] for the similar system parameters. Note that in the tone reservation method, some tones are reserved for the PAPR reduction and some of the tones are not used for transmitting, implying a loss in data rate.

In [35], another approach, similar to [12], is introduced for the SLM. The authors have introduced this method for MIMO-OFDM systems. The simulation results in [35] is similar to [12].

In [36], by extending the SLM method, a good PAPR reduction is achieved. A set of distinct sequences are generated from the data by using a modified repeat accumulate code. For a 128 channel OFDM system, employing QPSK modulation, a 2.75dB PAPR reduction close to 10^{-3} symbol clip rate is gained by using a three stage Linear Feedback Shift Register (LFSR). By defining a clipping ratio of two an additional 2dB reduction in the PAPR is achieved. However, our proposed methods based on the Hadamard constellation can lower the PAPR more (about 4dB by using only the hadamard constellation, and about 5dB or 6dB by using one or two redundant bits in the SLM method). Note that the method introduced in [36], similar to our proposed method, can recover the data in the receiver without prior knowledge of the selected sequence.

VI. CONCLUSION

We have proposed a constellation shaping method that achieves a substantial reduction in the PAPR in an OFDM system with a low complexity. An SLM technique is applied to this constellation to further reduce the PAPR of the OFDM signal. The proposed scheme significantly outperforms other PAPR reduction techniques reported in the literature . This technique offers a PAPR about 2dB to 3dB lower than that of some recent works, without the additional costs in energy and/or spectral efficiency. Moreover, it has a small computational complexity.

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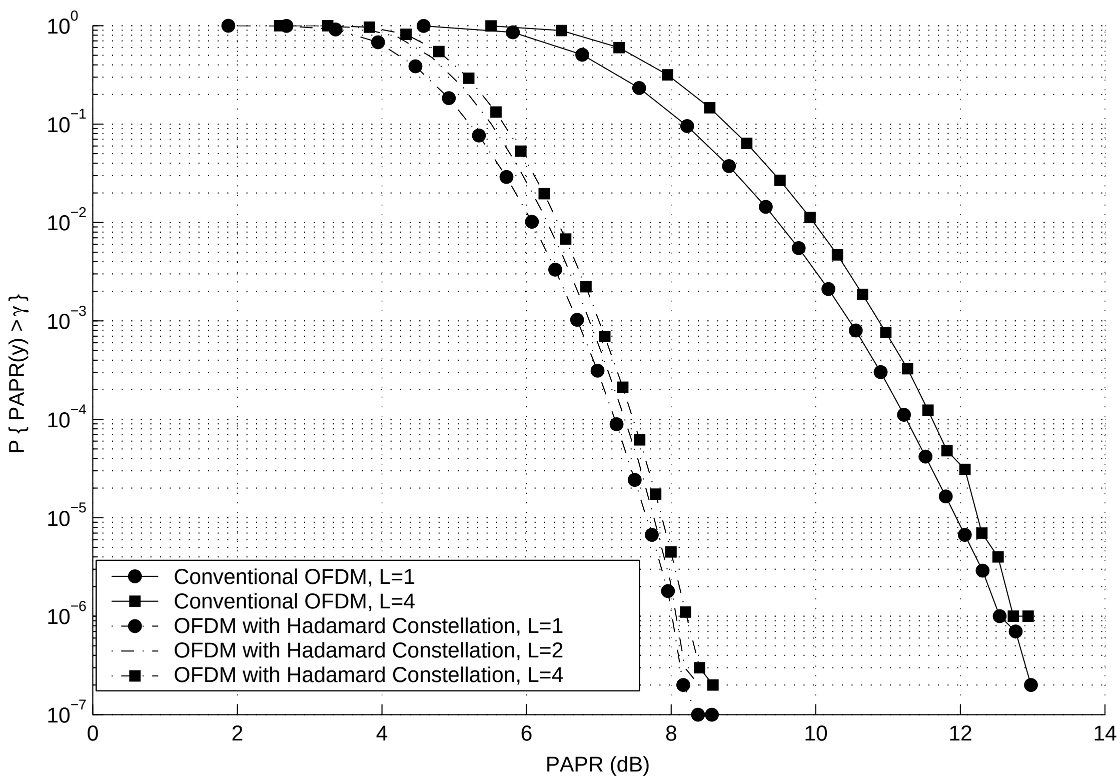

Fig. 4. CCDF of PAPR by using Hadamard constellation in a 128 channel OFDM system employing 16-QAM constellation with different oversampling factors for continuous PAPR approximation.

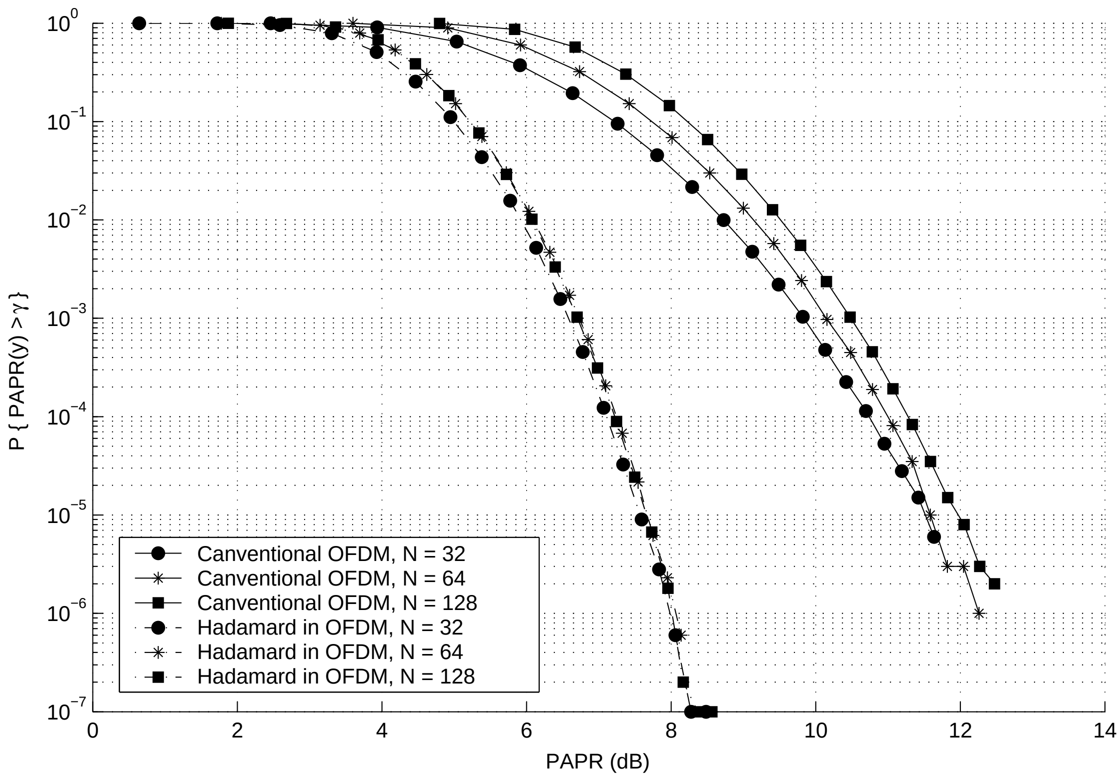

Fig. 5. CCDF of PAPR by using Hadamard constellation in an N channel OFDM system employing 16-QAM constellation.

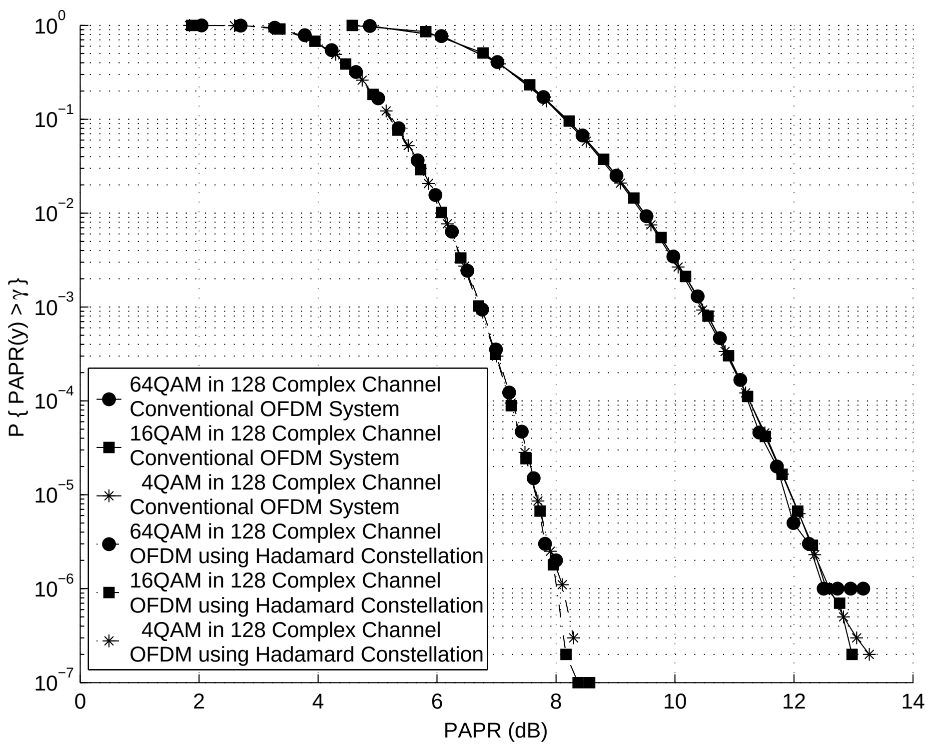

Fig. 6. CCDF of PAPR by using Hadamard constellation in a 128 channel OFDM system employing different QAM constellations.

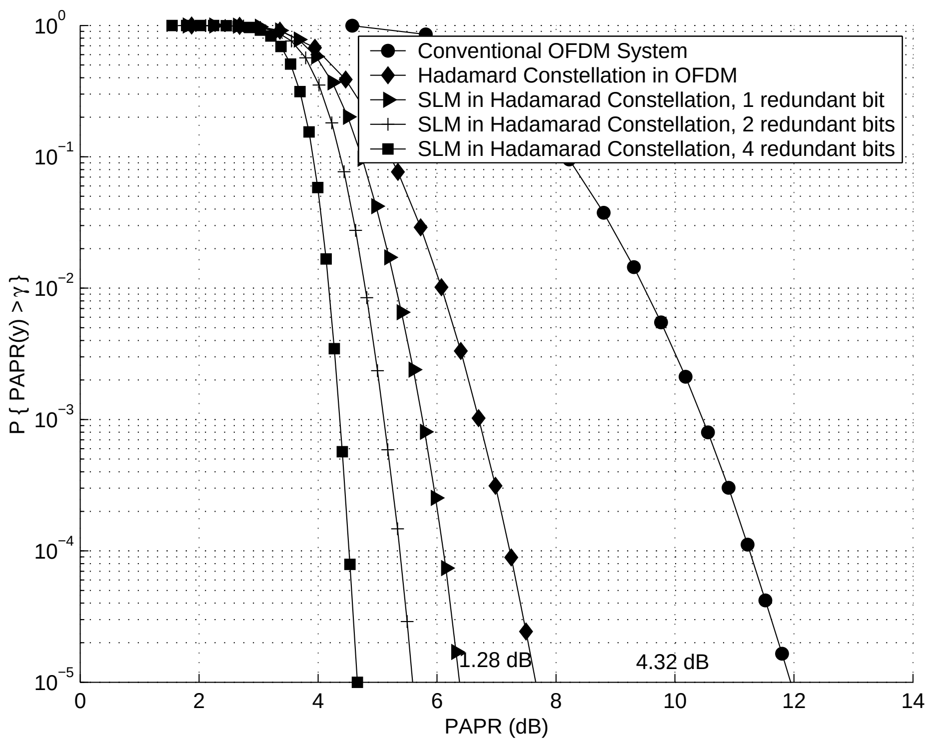

Fig. 7. CCDF of PAPR by SLM method based on Hadamard constellation in a 128 channel OFDM system employing 16-QAM constellation.

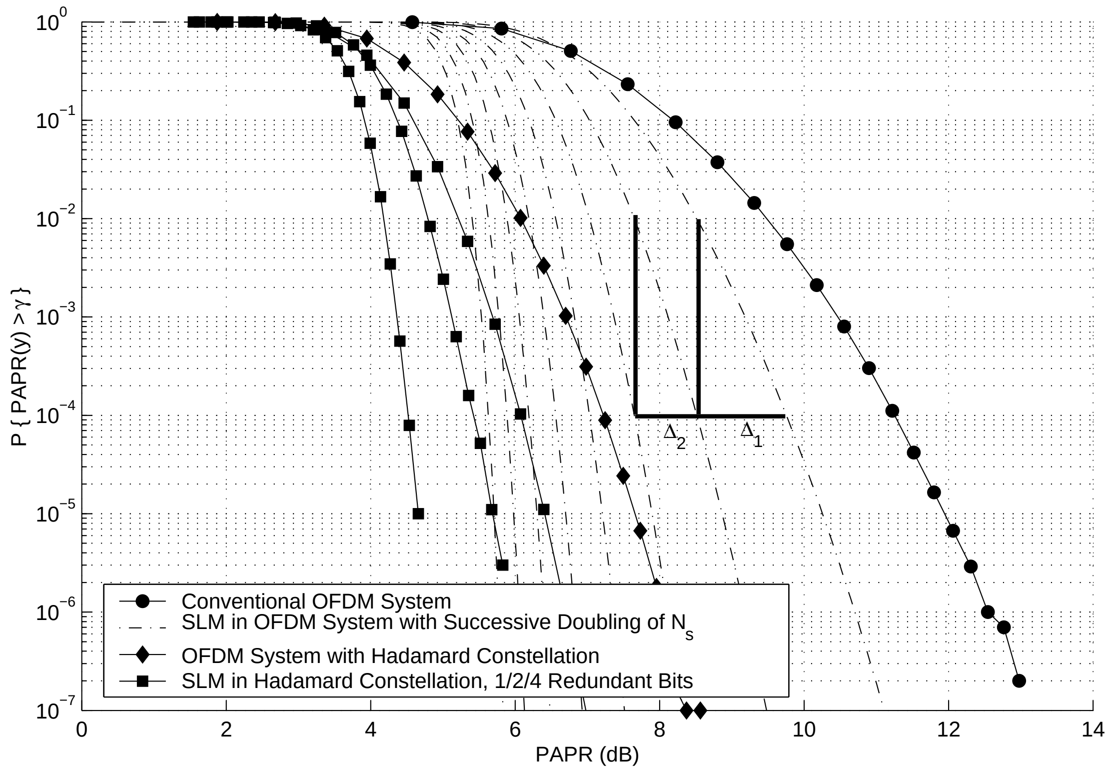

Fig. 8. CCDF of PAPR in a 128 channel OFDM system with SLM method using different number of redundant bits.