PAPR Reduction in OFDM Systems Using Constellation Shaping

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***Abstract*—This work considers the problem of Peak to Average Power Ratio (PAPR) reduction in an Orthogonal Frequency Division Multiplexing system. We design a cubic constellation, called Hadamard constellation, whose boundary is along the bases defined by the Hadamard matrix in the transform domain and then we further reduce the PAPR by applying the Selective Mapping technique. The encoding algorithm is based on a decomposition known as Smith Normal Form and has a minimal complexity. The proposed scheme significantly outperforms the other PAPR reduction techniques reported in the literature (offering about $2dB$ to $3dB$ lower PAPR compared to some recent techniques without any additional cost in terms of energy and/or spectral efficiency) and has a small computational complexity.**

I. Introduction

Orthogonal Frequency Division Multiplexing (OFDM) is a multicarrier transmission technique which is widely adopted in different communication applications. OFDM prevents Inter Symbol Interference (ISI) by inserting a guard interval and mitigates the frequency selectivity of a multi-path channel by using a simple equalizer. This simplifies the design of the receiver and leads to inexpensive hardware implementations. In systems with a high-speed transmission of data, large delay spreads can seriously limit the system performance due to ISI. Moreover, OFDM offers some advantages in higher order modulations and in the networking operation that position OFDM as the technique of choice for the next generation of wireless networks. However, OFDM systems have the undesirable feature of a large Peak to Average Power Ratio (PAPR) of the transmitted signals. Consequently to prevent the spectral growth of the OFDM signal, the transmit amplifier must operate in its linear regions. Therefore, power amplifiers with a large linear region are required for OFDM systems, but such amplifiers will continue to be a major cost component of OFDM systems. Consequently, reducing the PAPR is pivotal to reducing the expense of OFDM systems.

In this paper, we propound a constellation as a shaping method in an OFDM system with a low complex encoding method, based on [1], and a considerable PAPR reduction. A new Selective Mapping (SLM) method is applied in conjunction with our constellation to further reduce the PAPR in the OFDM signals.

II. Constellation Shaping

A large number of methods for PAPR reduction have been proposed. Constellation shaping is used in this paper to combat the PAPR problem in an OFDM system. In constellation shaping, a constellation in the frequency domain must be found such that the resulting shaping region in the time domain has a low PAPR. In a PAPR reduction problem, the peak value of the time domain signal vector $\mathbf{y}$ should be bounded by a specified value, $\|\mathbf{y}\|_\infty \leq \beta$. In an OFDM system the real time domain signal is related to the real frequency constellation point $\mathbf{x}$ by $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the real representation of the Inverse Fast Fourier Transform (IFFT) matrix. Therefore, this inequality on the time domain boundary translates to a parallelotope[1] in the frequency domain, defined by $\mathbf{A}^{-1}$. The points inside this parallelotope are used as constellation points in transmitting the OFDM signals. The principal challenge in constellation shaping is to find a unique way to map the input data to the constellation points such that the mapping (encoding) and its inverse (decoding) can be implemented by a reasonable complexity. In [1], the relations and generators in a free Abelian group in the integer domain are used so that the parallelotope corners lie in the integer lattice. Therefore, the constellation boundary is based on a parallelotope, defined by $\mathbf{Q}_N = \left[\alpha \mathbf{A}^{-1}\right]$, where $[.]$ represents rounding, and α is the smallest value to have the same constellation points as those of the unshaped constellation.

The encoding and decoding of the constellation points are derived from the decomposition of the matrix $\mathbf{Q}_N$ by column and row operations. Indeed, in the mathematical literature this decomposition is known as the Smith Normal Form (SNF) of an integer matrix [2]. SNF has been used in many applications in different fields, for example in solving linear diophantine equations, finding permutation equivalence and similarity of matrices, determining the canonical decomposition of finitely generated Abelian groups, integer programming, computing additional normal forms such as Frobenius and Jordan normal forms, and separable computation of Discrete Fourier Transform.

Theorem 1: Any integer matrix $\mathbf{Q}_N$ can be decomposed into $\mathbf{Q}_N = \mathbf{UDV}$, where $\mathbf{D}$ is diagonal with the entries $\{\sigma_i\}_{i=1}^N$ such that $\sigma_1 \mid \sigma_2 \mid \cdots \mid \sigma_N$, and $\mathbf{U}$ and $\mathbf{V}$ are unimodular matrices. The matrix $\mathbf{D}$ is called the SNF of the matrix $\mathbf{Q}_N$.

The complexity of this algorithm is the result of the computation of the SNF decomposition for the matrix $\mathbf{Q}_N$. We can use the SNF decomposition methods for the encoding procedure; however, the computational complexity for OFDM systems that are defined by the IFFT matrix remains very high.

[1]The parallelotope bases are defined along the columns of $\mathbf{A}^{-1}$.

III. HADAMARD CONSTELLATION IN AN OFDM SYSTEM

In [1], it is exhibited that if the matrix $\mathbf{Q}_N$ in OFDM multicarrier modulation can be changed by the Hadamard matrix, a very simple encoding algorithm will result. This simplicity is hidden in the recursive formula for the Hadamard matrix and its SNF decomposition [1]. However, this type of multicarrier modulation is not very popular because it does not offer all the advantages of conventional OFDM systems [3].

We consider this problem from another viewpoint. The constellation that should be used in an OFDM system has a boundary along the bases of the IFFT matrix, but the encoding of containing constellation points cannot be easily implemented. We propose a cubic constellation, called the Hadamard constellation, for an OFDM system whose boundary is along the bases defined by the Hadamard matrix in the transform domain. The IFFT and Hadamard matrices are both orthogonal matrices, and therefore, the constellations along these orthogonal bases are a rotated version of each other. This idea is illustrated in Fig. 1. By substituting the proper constellation along the IFFT matrix by the Hadamard matrix in an OFDM system, the resulting PAPR reduction is reduced; however, the encoding of this constellation, based on the SNF decomposition of the Hadamard matrix, is simple and practical. Due to the special recursive structure of the Hadamard matrix, its SNF decomposition can be easily expressed in a recursive format. Moreover, the encoding algorithm can be implemented by a butterfly structure that uses bit shifting and logical AND structures [1].

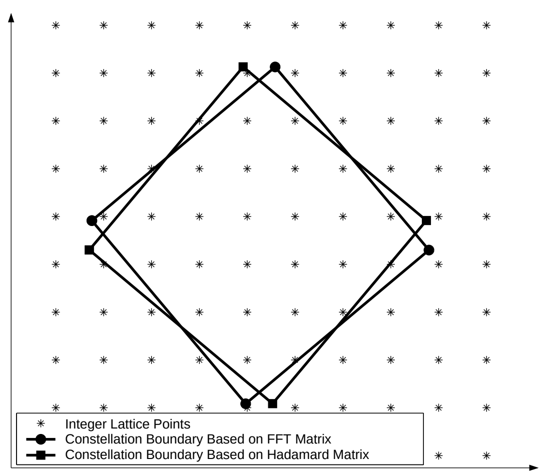

Fig. 1. N-D signal constellation for IFFT and Hadamard matrix.

The advantage of using the Hadamard constellation is not only a simple encoding algorithm with a low PAPR, but also the Hadamard constellation's ability to be concatenated with other methods. This motivates us to apply a SLM technique [4, 5] to the Hadamard constellation in an OFDM system. In typical SLM methods [4, 5], the major PAPR reduction is achieved by the first few redundant bits. Employing more redundant bits necessitates a high level of complexity to obtain modest improvements in the PAPR reduction. However, in the proposed SLM method, employing the Hadamard constellation causes a considerable PAPR reduction by itself. As a result, this method, by just using one or two redundant bits, significantly outperforms the other PAPR reduction techniques, reported in the literature [6].

In [6], we have investigated some issues that have emerged regarding the use of the Hadamard constellation in an OFDM system. In the encoding procedure, the cartesian boundary is employed [7] by viewing the real and imaginary parts of the signal as two separate real signals; that is, all the desired 2-D points are inside a square in the complex plane [6]. Therefore, all the constellation points inside a cubic boundary should be encoded.

If we represent the SNF decomposition of the Hadamard matrix by $\mathbf{H}_N = \mathbf{U}_N \mathbf{D}_N \mathbf{V}_N$, the encoding procedure operates in the following steps [1]:

$$\begin{aligned} \hat{\mathbf{x}} &= \mathbf{U}_N \boldsymbol{\lambda} \\ \boldsymbol{\gamma} &= \left\lfloor \frac{\mathbf{H}_N^T \hat{\mathbf{x}}}{N} \right\rfloor \\ \mathbf{x} &= \hat{\mathbf{x}} - \mathbf{H}_N \boldsymbol{\gamma} \\ \mathbf{y} &= \text{ IFFT } (\mathbf{x}), \end{aligned} \tag{1}$$

where $N = 2^n$, and $\boldsymbol{\lambda}$ is the canonical representation of integers I representing the constellation points $\mathbf{x}$. The canonical representation of any integer can be calculated by the recursive modulo operations; namely,

$$\begin{aligned} \lambda_1 &= I \bmod \sigma_1 \\ I_1 &= \frac{I - \lambda_1}{\sigma_1} \\ \lambda_i &= I_{i-1} \bmod \sigma_i \\ I_i &= \frac{I_{i-1} - \lambda_i}{\sigma_i}, \end{aligned} \tag{2}$$

where $1 \leq i \leq N$.
The reverse operation for finding I from the N-D vector $\mathbf{x}$ is

$$\begin{aligned} &\boldsymbol{\lambda} = \mathbf{U}_N^{-1} \mathbf{x} = (\lambda_1, \lambda_2, \cdots, \lambda_N)^T, \\ &\tilde{\lambda}_i = \lambda_i \bmod \sigma_i\ , \\ &I = \tilde{\lambda}_1 + \sigma_1(\tilde{\lambda}_2 + \sigma_2(\cdots(\tilde{\lambda}_{N-1} + \sigma_{N-1}\tilde{\lambda}_N)\cdots)). \end{aligned} \tag{3}$$

The number of points, inside the shaped constellation, is determined by the determinant of the Hadamard matrix.

Theorem 2: The constellation size for a $2^n \times 2^n$ Hadamard matrix is $\det(\mathbf{H}_{2^n}) = 2^{n2^{n-1}}$.
Therefore, the encoding algorithm necessitate the computation of the canonical representation of the large integer numbers in the transmission of the OFDM signals. The canonical representation can be simplified by using the fact that digital communication systems deal with binary input streams. Based on the SNF of the Hadamard matrix $\{\sigma_i\}_{i=1}^{2^n}$, the binary input data stream can be expressed as the canonical representation of the integers representing the Hadamard constellation points [6].

According to the Hadamard constellation size, the rate of the transmission is related to the number of subcarriers N in an OFDM system[2]. This rate is unacceptable not only because of the dependency on N but also because the rate usually is much higher than the required rate. A selection method for the constellation points must be implemented such that the constellation has the desired rate, and the time domain constellation

[2]For $N = 2^n$, the rate for each real component is $\log_2(2^{n2^{n-1}}/N) = \frac{n}{2}$.

points have a uniform distribution in the cubic constellation. A non-uniform distribution for the time domain constellation points leads to a higher PAPR in an OFDM system. In [6], it is exhibited that the selected integers representing the Hadamard constellation points should have a uniform distribution. The selection of N/a points among N integers will lead to the selection of the multiples of a. This scheme can control the constellation size, and consequently, the rate of transmission.

We also analyze the noise in different parts of the decoding procedure [6] and show the output noise that accompanies the canonical representation is a uniformly distributed noise; however, the noise that we receive along with the time domain constellation point is a gaussian noise and the maximum likelihood decoding algorithm for removing the noise is rounding off the constellation points in the integer domain.

IV. Selective Mapping along with Hadamard Constellation

In the following, we propose a SLM method [4,5] in conjunction with the hadamard constellation to further improve the performance . SLM is a method to reduce the PAPR in an OFDM system which involves generating a large set of data vectors that represent the same information, where the data vector with the lowest PAPR is used for the transmission. Assume that the data rate to be transmitted is r bits per block of length-N Fast Fourier Transform (FFT) symbol. Let r_s denote the number of redundant bits of r bits specified for SLM ($r_s << r$ and $r = \log_2(det(\text{constellation size})))$. Consequently, $N_s = 2^{r_s}$ constellation points should represent the same information. In this method, the input integers I are mapped to the Hadamard constellation points, and the output integers are classified by the sets with the same r_s Most Significant Bits (MSBs). All the corresponding constellation points in each set represent the same information. The IFFT operation for all these constellation points in each set is computed, and the constellation point with the lowest PAPR is transmitted.

As mentioned before, the Hadamard constellation is a rotated version of a cubic constellation, so the average energy and spectral efficiency of this constellation is the same as those in a cubic constellation. However, in Selective Mapping method we have a slight decrease in the average energy, since we choose the point with the lowest PAPR among a few number of possible points in the Hadamard constellation.

Fig. 2 shows the simulation result of our PAPR reduction method for a 128 complex channel OFDM system employing 16-QAM constellation. We compare our proposed method by some recent methods reported in the literature [8–11]. We can lower PAPR as compared to these methods (about $3dB$ lower PAPR as compared to [8–10] and about $2dB$ lower PAPR as compared to [11]). Moreover, unlike other SLM methods, no side information is required. Proposed SLM method can obtain lower PAPR reduction by using fewer redundant bits, because this method is applied to the Hadamard constellation in OFDM systems and using this constellation has a relatively low PAPR by itself. The complexity of the Hadamard constellation along with SLM is very low and the overall scheme is practically feasible (it competes in terms of the overall complexity with the alternative techniques reported in the literature).

V. Conclusion

We propose a constellation shaping method that achieves a substantial reduction in the PAPR in an OFDM system with a low complexity. An SLM technique is applied to this constellation to further reduce the PAPR of the OFDM signal. The proposed scheme significantly outperforms other PAPR reduction techniques reported in the literature (Our scheme offers about a $2dB$ to $3dB$ lower PAPR, compared to the best known techniques, without the additional costs in energy and/or spectral efficiency), and has a small computational complexity.

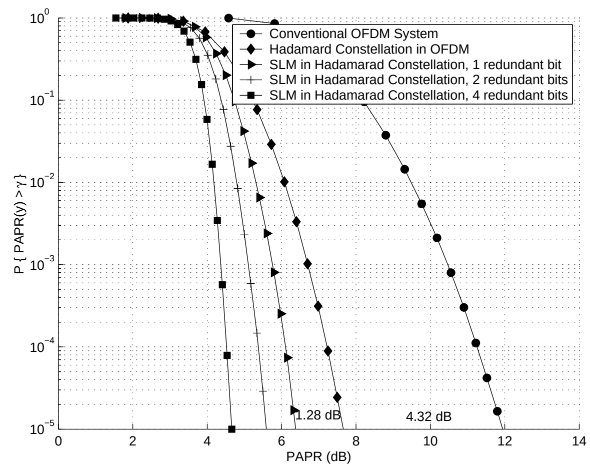

Fig. 2. Probability of PAPR in a 128 channel OFDM system with 16-QAM constellation employing Hadamard constellation and Selective Mapping.

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