Question 4 (6 marks) Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Bin}(n, p)$.

(a) [3] Evaluate $E[3X + 2Y]$ (simplify your answer).

(b) [3] Evaluate $E[X! + e^X]$ (simplify your answer). What are the conditions on λ to have a finite expectation?

Question 5 (6 marks) A transmitter in a communication system consists of 5 antenna arranged in a line. The transmitter will be functional if no two consecutive antennas are defective. Calculate the probability that the transmitter is functional if

(a) [2] Exactly two of the five antennas are defective (assuming all configurations are equally likely).

(b) [4] Each antenna is defective with probability $\frac{1}{2}$, independent of each other.

Q.4 : a) $E(X) = \lambda$, $E(Y) = np$

$$E(3X+2Y) = 3E(X) + 2E(Y) = 3\lambda + 2np$$

b) $E(X! + e^X) = E(X!) + E(e^X)$

$$E(X!) = \sum_{K=0}^{\infty} \cancel{(K!)} \frac{e^{-\lambda} \cdot \lambda^K}{\cancel{K!}} = e^{-\lambda} \cdot \sum_{K=0}^{\infty} \lambda^K = e^{-\lambda} \cdot \frac{1}{1-\lambda}, \text{ if } |\lambda| < 1 .$$

$$E(e^X) = \sum_{K=0}^{\infty} (e^K) \cdot \frac{e^{-\lambda} \cdot \lambda^K}{K!} = e^{-\lambda} \cdot \sum_{K=0}^{\infty} \frac{(\lambda e)^K}{K!} = e^{-\lambda} \cdot e^{(\lambda e)}$$

Q.5 : a)  $\longrightarrow P = \dfrac{\binom{4}{2}}{\binom{5}{2}} = \dfrac{6}{10} = \dfrac{3}{5}$

b) F = functional D = # of defective antennas

$$p(F) = p(F|D=0)p(D=0) + p(F|D=1)p(D=1) + \cdots + p(F|D=5)p(D=5)$$

$\star$ $p(F|D=4) = 0$, $p(F|D=5) = 0$, $p(D=k) = \binom{5}{k}(1/2)^k(1/2)^{5-k}$

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$p(F|D=0) = 1$, $p(D=0) = \binom{5}{0}(1/2)^0(1/2)^5 = \frac{1}{32}$

$p(F|D=1) = 1$, $p(D=1) = \binom{5}{1}(1/2)^1(1/2)^4 = \frac{5}{32}$

$p(F|D=2) = \frac{3}{5}$ (part "a") , $p(D=2) = \binom{5}{2}(1/2)^2(1/2)^3 = \frac{5}{16}$

$p(F|D=3) = \frac{1}{\binom{5}{3}} = \frac{1}{10}$, $p(D=3) = \binom{5}{3}(1/2)^3(1/2)^2 = \frac{5}{16}$

$$\longrightarrow p(F) = \frac{13}{32}$$

"Textbook: exercise 4.12 of ch.4 and example 4.c of ch.1."

Question 3: A communication system consists of n components, each of which will, independently, function with probability p. Let X be the number of the components that function.

(5/40) a: What is the probability mass function of X? Compute $E(X)$, $Var(X)$, $E(2X+3)$ and $Var(2X+3)$.

(5/40) b: Suppose that the system will be functional if more than one-half of its components function. We denote Y a random variable which takes value 1 if the system functions and value 0 if it does not function. What is the probability mass function of Y if $n = 6$ and $p = 2/3$?

Question 4: Solve the following problems:

(5/40) a: The average number of database queries processed by a computer in any 10-second interval is 5 (hint: use Poisson distribution). What is the probability that there will be no queries processed in a 10-second interval? What is the probability that at least two queries are processed during this time?

3. Assume that components will function independently. Let X be the number of components that are functional. Then X can be considered as a binomial random variable.
(a)

the p.m.f of X :

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, \cdots, n$$

$$E[X] = np \text{ and } Var(X) = np(1-p)$$

$$E[2X+3] = 2np + 3 \text{ and } Var(2X+3) = 4np(1-p)$$

(b) (3 marks)

$$\begin{aligned} q &= P\{X > 3\} \\ &= P\{X = 4\} + P\{X = 5\} + P\{X = 6\} \\ &= 15(2/3)^4(1/3)^2 + 6(2/3)^5(1/3) + (2/3)^6 \\ &= 2^4 \times 31/3^6 = .6804 \end{aligned}$$

The p.m.f of Y: (2 marks)

$$P\{Y = 1\} = q = .6804 \text{ and } P\{Y = 0\} = 1 - q = .3196$$

4. (a)
(2.5 marks) Let X be the number of database queries by a computer in a 10-second interval. Then we can consider X being a Poisson random variable with parameter $\lambda = 5$.

(2.5 marks) The desire probabilities can be computed as follows.

$$P\{X = 0\} = e^{-5} = 0.0067$$

$$P\{X \geq 2\} = 1 - P\{X < 2\} = 1 - e^{-5} - 5e^{-5} = .9596$$

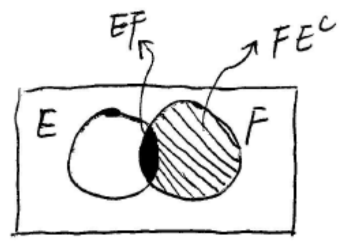
6. Suppose that n independent trials, each of which results in any of the outcomes 0, 1, or 2 with respective probabilities, p_0, p_1, and p_2, $\sum_{i=0}^{2} p_i = 1$, are performed. Find the probability that outcome 1 occurs at least once and outcome 2 occurs exactly twice.

#6. Let $E = \{$ outcome 1 occurs at least once $\}$

$F = \{$ outcome 2 occurs exactly twice $\}$.

then the desired probability is actually $P(EF)$.

Since $P(EF) = P(F) - P(FE^c)$, now it's required to compute the probability $P(F)$ and $P(FE^c)$ respectively.



$P(F) = \binom{n}{2} P_2^2 (1-P_2)^{n-2}$,

$P(FE^c) = P\{$ outcome 2 occurs twice and outcome 1 never occurs $\}$ $\therefore P(EF) = P(F) - P(FE^c)$

$= \binom{n}{2} P_2^2 P_0^{n-2}$ ($\because$ Except the twice outcome 2, all trials result in outcome 0)

Therefore, $P(EF) = P(F) - P(FE^c) = \binom{n}{2} P_2^2 (1-P_2)^{n-2} - \binom{n}{2} P_2^2 P_0^{n-2}$

4) In a batch of manufactured units, 2% of the units have the wrong weight (and perhaps also the wrong color), 5% have the wrong color (and perhaps also the wrong weight), and 1% have both the wrong weight and the wrong color.

i) A unit is taken at random from the batch. What is the probability that the unit is defective in at least one of the two respects? (10 points)

ii) What is the probability mass function of the number of units selected to find the first non-defective item? (10 points)

Q4) i) W = the unit has the wrong weight (may be the wrong colar) and

C = " " " " " Colar (and maybe the wrong w)

$W \cap C$ = both wrong weight and wrong Colar

$p(W) = 0.02$, $p(C) = 0.05$, $p(W \cap C) = 0.01$

p{ the unit is defective in at least one respect }

$= p\{W \cup C\} = p\{W\} + p\{C\} - p\{W \cap C\} = 0.02 + 0.05 - 0.01$

$= \boxed{0.06}$

ii) X = # of units selected to find the first non-defective item

$p(X=k) \triangleq$ the k^{th} item is non-defective and the previous $(k-1)$ items are defective

"X" follows Geometric distn. with $p = 1 - 0.06 = 0.94$

$\longrightarrow p(X=k) = (1-p)^{k-1} \cdot p$, $k = 1, 2, 3, \ldots$

$= (0.06)^{k-1} \cdot (0.94)$, $k = 1, 2, 3, \ldots$

4. (15 points) A newsboy purchases papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10$ and $p = 1/3$, approximately how many papers should he purchased so as to maximized his expected profit?

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Let random variable D denote the daily demand and random variable X_m denote the profit when the newsboy purchases m papers $(m \leq 10)$. We note that:

$$X_m = \begin{cases} m(15-10) & D \geqslant m \\ D(15-10)+(m-D)(-10) & D < m \end{cases}$$

The random variable X_m is a function of the random variable D. That is $X_m = g(D)$ where

$$g(x) = \begin{cases} 5m & x \geqslant m \\ 15x - 10m & x < m \end{cases}$$

Therefore,

$$E[X_m] = \sum_{d=0}^{10} g(d)\, p\{D=d\}$$

$$= \sum_{d=0}^{m-1} (15d-10m)\binom{10}{d}(1/3)^d (2/3)^{10-d}$$

$$+ \sum_{d=m}^{10} (5m)\binom{10}{d}(1/3)^d (2/3)^{10-d}$$

$\Rightarrow$ $E[X_0] = 0$, $E[X_1] = 4.74$, $E[X_2] = 8.18$, $E[X_3] = 8.69$

$E[X_4] = 5.3$, $E[X_5] = -1.5$, ... $\Rightarrow$ $\{m_{opt} = 3\}$

$$E\{X_m\} = \sum_{d=0}^{m-1}(15d-10m)\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} + \sum_{d=m}^{10}(5m)\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d}$$

$$= \sum_{d=0}^{m-1}(15d-15m)\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} + \underbrace{5m\sum_{d=0}^{m-1}\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} + 5m\sum_{d=m}^{10}\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d}}_{5m\sum_{d=0}^{10}\binom{10}{d}\left(\frac{1}{3}\right)^d\left(\frac{2}{3}\right)^{10-d}}$$

$$\downarrow$$

$$15\sum_{d=0}^{m-1}(d-m)\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} + 5m\times 1$$

For finding recursive formula:

$$E\{X_{m+1}\} - E\{X_m\} = 15\left\{\sum_{d=0}^{m}(d-m-1)\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} - \sum_{d=0}^{m-1}(d-m)\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d}\right\}$$

Same

$$+5(m+1-m)$$

$$E\{X_{m+1}\} - E\{X_m\} = 15\left(\underbrace{(m-m)}_{0}\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d}\right) - 15\sum_{d=0}^{m}\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} + 5$$

$m=0 \Rightarrow E\{X_0\}=0$

$$E\{X_1\} = -15\sum_{d=0}^{0}\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} + 5 + \overset{E\{x_0\}}{0} = 5 - 15\times\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^{10} = 4.74$$

$$E\{X_2\} = E\{X_1\} + 5 - 15\sum_{d=0}^{1}\binom{10}{d}\left(\tfrac{1}{3}\right)^d\left(\tfrac{2}{3}\right)^{10-d} = 4.74 + 5 - 15\left\{\underbrace{\left(\tfrac{2}{3}\right)^{10} + 10\times\left(\tfrac{1}{3}\right)^1\left(\tfrac{2}{3}\right)^9}_{0.1040}\right\} = 8.18$$

Maximum

$$E\{X_3\} = 8.18 + 5 - 15\Big[\underbrace{0.1040 + \binom{10}{2}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^8}_{0.299}\Big] = 8.69 \searrow E\{X_4\} = 5.3$$

Q.1: when Coin 1 is flipped, it lands heads with prob. 0.4, when Coin 2 is flipped, it lands heads with prob. 0.7. One of these Coins is randomly chosen and flipped 10 times.

a) what is the prob. that exactly 7 of 10 flips land on heads?

$$P(7 \text{ heads}) = P(7 \text{ heads} \mid \text{Coin } 1 \text{ is selected}) P(\text{Coin } 1 \text{ is selected}) + P(7 \text{ heads} \mid \text{Coin } 2 \text{ is selected}) P(\text{Coin } 2 \text{ is selected})$$

using the Binomial distn function, $\binom{n}{k} p^k (1-p)^{n-k}$, we have

$$P(7 \text{ heads}) = \binom{10}{7}(0.4)^7(0.6)^3\left[\tfrac{1}{2}\right] + \binom{10}{7}(0.7)^7(0.3)^3\left[\tfrac{1}{2}\right]$$

b) Given that the first one is head, what is the conditional prob. that exactly 7 of 10 flips land on heads?

$$P(7 \text{ heads} \mid 1^{st} \text{ flip is heads}) = \frac{P(7 \text{ heads}, 1^{st} \text{ flip is heads})}{P(1^{st} \text{ is heads})}$$

$$= \frac{P(7 \text{ heads}, 1^{st} \text{ flip is heads} \mid \text{Coin } 1 \text{ is selected}) P(\text{Coin } 1) + P(7 \text{ heads}, 1^{st} \text{ is heads} \mid \text{Coin } 2) P(\text{Coin } 2)}{P(1^{st} \text{ is heads} \mid \text{Coin } 1) P(\text{Coin } 1) + P(1^{st} \text{ is heads} \mid \text{Coin } 2) P(\text{Coin } 2)}$$

$$= \frac{\left[\binom{9}{6}(0.4)^6(0.6)^3(0.4)\right]\left[\tfrac{1}{2}\right] + \left[\left(\binom{9}{6}(0.7)^6(0.3)^3\right)(0.7)\right]\left[\tfrac{1}{2}\right]}{(0.4)(\tfrac{1}{2}) + (0.7)(\tfrac{1}{2})}$$

Q.2 : Approximately 80,000. marriages have been recorded in NYC last year. Estimate the prob. that for **at least** one of these couples :

a) Both partners were born on April $\underline{30}^{th}$?

- Binomial distn: $\binom{n}{k} p^k (1-p)^k$
- Poisson distn: $p(X=k) = \dfrac{e^{-\lambda} \cdot \lambda^k}{k!}$
- when $\begin{cases} n \text{ is large } (i.e., n \geq 30) \\ p \text{ is small } (i.e., p \to 0) \end{cases}$, we can approximate "Binomial distn", using poisson distn with $\lambda = np$, i.e., $\binom{n}{k} p^k (1-p)^k \cong \dfrac{e^{-(np)} (np)^k}{k!}$.

$P =$ prob. that both partner (for an arbitrary couple) were born on April. 30^{th} $= (\frac{1}{365})(\frac{1}{365}) = (\frac{1}{365})^2$

Therefore, $\begin{cases} n = 80{,}000 \text{ (large)} \\ P = (\frac{1}{365})^2 \text{ (small)} \\ \lambda = np = \frac{80{,}000}{(365)^2} \cong 0.6 \end{cases} \mapsto$ desired prob. $= p\{X \geq 1\} = 1 - p\{X = 0\} = 1 - e^{-0.6}$

where $X = \#$ of ~~partners~~ ~~who~~ couples, both partners were born on Apr. 30^{th}.

b) Both partners were born on the same day of the year?

$P =$ prob. that both partners were born on the same day of the year $= \frac{1}{365}$

$\begin{cases} n = 80{,}000 \text{ (large)} \\ P = \frac{1}{365} \end{cases} \mapsto \lambda = np = \frac{80{,}000}{365} \cong 219.18$

$\to p\{X \geq 1\} = 1 - p\{X = 0\} = 1 - e^{-219.18}$

$X = \#$ of couples who both partners were born on the same day of the year.

*Note : We could solve this question using the regular "Binomial distn"; However, this example shows how and when we approximate "Binomial" using Poisson.