session 9

Solutions.

Q1. we try to find two events like A & B such that:

$P(X\in A, Y\in B) \neq P(X\in A)P(Y\in B)$.

consider $A = [0.5, 0.75]$

$B = [0.75, +\infty)$

$P(X\in A, Y\in B) = P(0.5 \leqslant X \leqslant 0.75, Y \geqslant 0.75) =$

$P(0.5 \leqslant X \leqslant 0.75, 1-X \geqslant 0.75) =$

$P(0.5 \leqslant X \leqslant 0.75, X \leqslant 0.25) = 0$

but,

$P(X\in A).P(Y\in B) = P(0.5 \leqslant X \leqslant 0.75)P(X \leqslant 0.25) = \frac{1}{4}\cdot\frac{1}{4}$

so they are dependent.

Q2) $f_{X_1}(x_1) = 1 \; ; \; 0 \leqslant x_1 \leqslant 1$

$f_{X_2}(x_2) = 1 \; ; \; 0 \leqslant x_2 \leqslant 1$

$f_{X_3}(x_3) = 1 \; ; \; 0 \leqslant x_3 \leqslant 1$

and because they are indep.

$\Rightarrow f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3)$

we should consider three possibilites
$\begin{cases} ① \; X_1 \text{ is the largest } \& \; X_1 > X_2+X_3 \\ ② \; X_2 \;\; " \;\; " \;\; " \;\; \& \; X_2 > X_1+X_3 \\ ③ \; X_3 \;\; " \;\; " \;\; " \;\; \& \; X_3 > X_1+X_2 \end{cases}$

desired prob. $= P(X_1 > X_2+X_3) + P(X_2 > X_1+X_3) + P(X_3 > X_1+X_2)$

continued

but because of the symmetric of the question over $X_1, X_2, X_3 \Rightarrow$
$P(X_1 > X_2+X_3) = P(X_2 > X_1+X_3) = P(X_3 > X_1+X_2)$.

$$\Rightarrow \text{Ans} = 3 \cdot P(X_1 > X_2+X_3) = 3 \cdot \iiint_{0 \leq X_2+X_3 < X_1 \leq 1} f_{X_1,X_2,X_3}(x_1,x_2,x_3)\,dx_1 dx_2 dx_3$$

$$= 3 \cdot \int_0^1 \int_0^{1-X_3} \int_{X_2+X_3}^1 dx_1 dx_2 dx_3 = 3 \cdot \int_0^1 \int_0^{1-X_3} (1-X_2-X_3)\,dx_2 dx_3 =$$

$$3 \cdot \int_0^1 \frac{(1-x_3)^2}{2} dx_3 = \frac{1}{2}.$$

To practice let's change the order of x_1, x_2 and x_3 to see how the boundries are changed. (orders in integral)

$$\text{Ans} = 3 \cdot \int_0^1 \int_0^{X_1} \int_0^{X_1-X_3} dX_2\, dX_3\, dX_1 = 3 \cdot \int_0^1 \int_0^{X_1} (X_1 - X_3)\, dX_3 dX_1$$

$$= 3 \int_0^1 \frac{X_1^2}{2} = 3 \cdot \frac{X_1^3}{6} \Big|_0^1 = \frac{1}{2}.$$

★ Here's the point: if the order of integrals are for example $dX_1 dX_2 dX_3 \dots dX_n$, we should find boundries for X_1 respect to $X_2, X_3 \cdots X_n$

boundries for X_2 respect to $X_3, X_4, \cdots, X_n$

$\vdots$

boundries for X_{n-1} respect to X_n

and boundries for X_n respect to actual boundry of X_n in the question.

Q3) $\int_0^1 \int_0^2 f_{X,Y}(x,y)\, dy\, dx = 1$

$$\Rightarrow \int_0^1 \int_0^2 c\left(x^2 + \frac{xy}{2}\right) dy\, dx = c\int_0^1 \left(x^2 y + \frac{xy^2}{4}\Big|_0^2\right) dx =$$

$$c\int_0^1 (2x^2 + x)\, dx = c.\left(\frac{2x^3}{3} + \frac{x^2}{2}\Big|_0^1\right) = c.\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{7}{6}c = 1$$

$$\Rightarrow c = \frac{6}{7}$$

b) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7}(2x^2 + x).$

c) $P\{X > Y\} = \iint_{X>Y>0} f_{X,Y}(x,y)\, dx\, dy = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\, dx =$

$$\frac{6}{7}\int_0^1 \left(x^3 + \frac{x^3}{4}\right) dx = \frac{6}{7}\left(\frac{x^4}{4} + \frac{x^4}{16}\right)\Big|_0^1 = \frac{6}{7}\cdot\left(\frac{1}{4} + \frac{1}{16}\right) = \frac{15}{56}.$$

d) $P\{Y > \frac{1}{2} \mid X < \frac{1}{2}\} = \dfrac{P\{Y > \frac{1}{2}, X < \frac{1}{2}\}}{P\{X < \frac{1}{2}\}} = \dfrac{\frac{6}{7}\int_{1/2}^{2}\int_0^{1/2} \left(x^2 + \frac{xy}{2}\right) dx\, dy}{\frac{6}{7}\int_0^{1/2} (2x^2 + x)\, dx} = \dfrac{69}{80}.$

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Q4. a) $P(y<1 \mid x<1) = \dfrac{P(y<1, x<1)}{P(x<1)}$

$$P(y<1, x<1) = \int_0^1 \int_x^1 2e^{-(x+y)}\,dy\,dx =$$

$$\int_0^1 2e^{-x} \cdot (-e^{-y})\Big|_x^1 dx =$$

$$\int_0^1 2e^{-x}(e^{-x} - e^{-1})\,dx = e^{-2} - 2e^{-1} + 1 = (1-e^{-1})^2$$

$$P(X<1) = \int_0^1 \int_x^\infty 2e^{-(x+y)}\,dy\,dx = \int_0^1 2e^{-x}(-e^{-y})\Big|_x^\infty dx$$

$$= \int_0^1 2e^{-x} e^{-x}\,dx = -e^{-2x}\Big|_0^1 = 1-e^{-2}$$

$$\Rightarrow P(y<1 \mid x<1) = \frac{(1-e^{-1})^2}{1-e^{-2}}$$

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b) $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$$f_X(x) = \int_x^\infty 2e^{-(x+y)}\,dy = 2e^{-2x}, \quad x>0$$

$$\Rightarrow f_{Y|X}(y|x) = \frac{2e^{-(x+y)}}{2e^{-2x}} = e^{-(y-x)} \qquad y>x>0$$

$$\Rightarrow P(y<1 \mid X=0.5) = \int_{0.5}^1 e^{-(y-0.5)}\,dy = 1-e^{-0.5}$$

$P(0.25 < Y < 0.75 \mid X=1) = 0$

Q5) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-y}/y}{\int_0^y e^{-y}/y \, dx} = 1/y$

$$\Longrightarrow f_{X|Y}(x|y) = \begin{cases} 1/y & ; \quad 0<x<y \quad , 0<y<\infty \\ 0 & ; \quad o.w \end{cases}$$

Simple question :-)

Q6) $f_{X_1}(x_1) = \frac{1}{L} \qquad 0 \leq x_1 \leq L$

$f_{X_2}(x_2) = \frac{1}{L} \qquad 0 \leq x_2 \leq L$

$f_{X_3}(x_3) = \frac{1}{L} \qquad 0 \leq x_3 \leq L$

we have two possibilities
(I) $P(0 \leq X_1 < X_2 < X_3 \leq L)$
(II) $P(0 \leq X_3 < X_2 < X_1 \leq L)$

(I) we try to find $P(0 \leq X_1 < X_2 < X_3 \leq L)$

$$Ans = \int_0^L \int_{x_1}^L \int_{x_1}^{x_3} \frac{1}{L}\frac{1}{L}\frac{1}{L} \, dx_2 \, dx_3 \, dx_1 =$$

$$\left(\frac{1}{L}\right)^3 \int_0^L \int_{x_1}^L (x_3 - x_1) \, dx_3 \, dx_1 =$$

$$\left(\frac{1}{L}\right)^3 \int_0^L \left(\frac{L^2}{2} - x_1 L - \frac{x_1^2}{2} + x_1^2\right) dx_1 = \left(\frac{1}{L}\right)^3 \int_0^L \left(\frac{L^2}{2} - x_1 L + \frac{x_1^2}{2}\right) dx_1 =$$

$$= \left(\frac{1}{L}\right)^3 \left(\frac{L^3}{2} - \frac{L^3}{2} + \frac{L^3}{6}\right) = \frac{1}{6}.$$

two valid poss.

$$\text{Answer of question} = P(0 \leq x_1 < x_2 < x_3 \leq L) + P(0 \leq x_3 < x_2 \leq x_1 \leq L)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}. \qquad \text{(because of symmetry)}$$

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