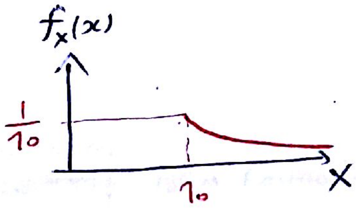
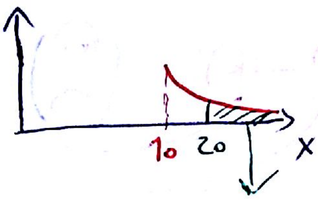
Problem 1:

$$f_X(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leqslant 10 \end{cases}$$



a) $P(X > 20)$



This area $= P(X > 20) = P(X \geqslant 20)$

In contrast with discrete random variable in continous case it is not important $\geqslant \equiv >$

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}\, dx = 10 \left(-\frac{1}{x}\right)\Big|_{20}^{\infty} = 10\left(0 - \left(-\frac{1}{20}\right)\right) = \boxed{\frac{1}{2}}$$

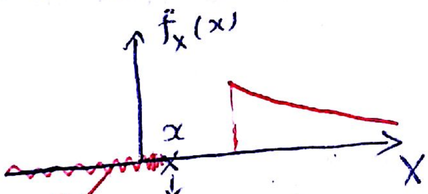
b) CDF $\triangleq P(X \leqslant x) = F(x)$

$$F(x) = \int_{-\infty}^{x} f_X(u)\, du$$

Dummy variable

= Since $f_X(x)$ has two segments CDF also has two segments

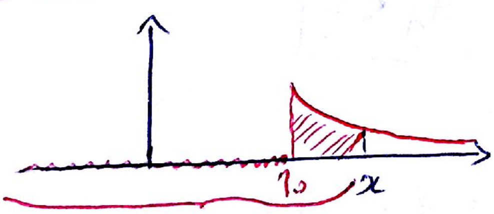
$x \leqslant 10 \longrightarrow$



for example

This area is CDF so: $F(x) = \int_{-\infty}^{x} 0\, dx = \boxed{0}$ if $x \leqslant 10$

$x > 10 \longrightarrow$



This area is CDF value in point $\underline{\underline{x}}$

$$F(x) = \int_{-\infty}^{10} 0 \cdot du + \int_{10}^{x} \frac{10}{u^2}\, du$$

$$= 10\left(-\frac{1}{u}\right)\Big|_{10}^{x} = 10\left(-\frac{1}{x} + \frac{1}{10}\right)$$

$$= \boxed{1 - \frac{10}{x}} \quad \text{if } x > 10$$

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d) what is the probability that among 6 devices at least 3 will function for at least 15 hours

$6 \downarrow n$, $3 \downarrow K$

quality

It is combination of binomial and ~~[illegible]~~ mentioned continous r.v. :

Binomial:

$$P(\# \text{ of } \oslash = k) = \binom{n}{k} P_{\oslash}^{k} (1-P_{\oslash})^{n-k}$$

We only need to find $P_{\oslash}$ which is: $P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2}\, dx = -10\left(0-\left(-\frac{10}{15}\right)\right) = \boxed{\frac{2}{3}}$

$\downarrow$ At least 15 hours

Note: We also could use CDF: $P(X \geq 15) = 1 - P(X < 15) = 1 - P(X \leq 15) = 1 - F(15)$

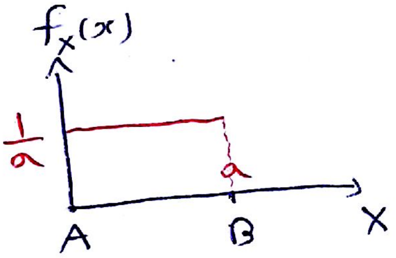
$= 1 - \left(1 - \frac{10}{15}\right) = \boxed{\frac{2}{3}}$

Problem 2:

Lands at a random point $\equiv$ probability of landing at all points are same

$\Downarrow$

Uniform distributed r.v



a) $\begin{cases} d_A < d_B \\ d_A + d_B = a \end{cases} \Rightarrow d_A + d_A < \underbrace{d_B + d_A}_{a} \Rightarrow \boxed{d_A < \frac{a}{2}}$

$$P\left(d_A < \frac{a}{2}\right) = \int_{-\infty}^{\frac{a}{2}} f_X(x)\, dx = \int_{0}^{\frac{a}{2}} \frac{1}{a}\, dx = \boxed{\frac{1}{2}}$$

b) $\begin{cases} d_A > 3d_B \\ d_A + d_B = a \end{cases} \Rightarrow d_A + 3d_A > 3(d_A + d_B) \Rightarrow d_A > \frac{3a}{4} \Rightarrow P\left(d_A > \frac{3a}{4}\right) = \int_{\frac{3a}{4}}^{a} \frac{1}{a}\, dx = \boxed{\frac{1}{4}}$

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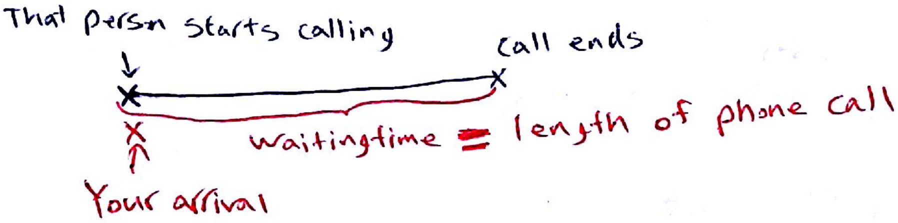
Problem 3:

$$X \sim \exp(0.1) \Rightarrow f_X(x) = 0.1\, e^{-0.1x} \quad x \geq 0$$

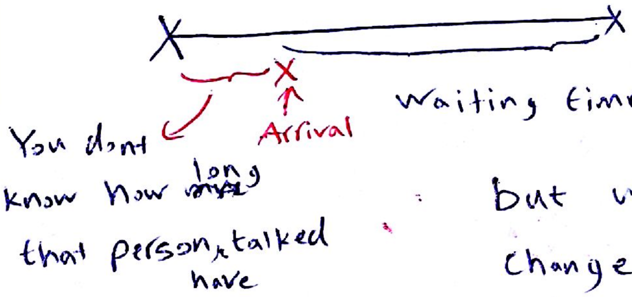
X ↓ length of phone call



a) $P(X>10) = \int_{10}^{\infty} 0.1\, e^{-0.1x}\, dx = -e^{-0.1x}\Big|_{10}^{\infty} = e^{-0.1(10)} = \boxed{e^{-1}}$

b) $P(10<X<20) = \int_{10}^{20} 0.1\, e^{-0.1x}\, dx = \left(-e^{-0.1(20)} - \left(-e^{-0.1(10)}\right)\right) = \boxed{e^{-1} - e^{-2}}$

Note: If your arrival time and beginning of that phone call was not at same time, the result does not change! This magical property is called memoryless and exponential distribution is the one and the only distribution which has this property



waiting time $\neq$ length of phone call

but waiting time $\sim \exp(0.1)$ so the result doesnt change

If you are confused do not read next lines.

Suppose that your arrival time is also an exponential with λ_x.

$P(\text{waiting time} > 10 \mid \text{when you arrived that person is talking}) = \boxed{e^{-1}}$

but $P(\text{waiting time} > 10) \neq e^{-1}$ it can be computed

$$\text{CDF of waiting time in this case} = \left(1 - e^{-0.1x}\right) . P(\text{Arrival time} < \text{Call length}) +$$

$$1 . P(\text{"} > \text{"})$$

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Problem 4: $X \sim \exp(0.2)$

$E\{X\} = \frac{1}{\lambda} = \frac{1}{0.2} = 5 \qquad Var\{X\} = \frac{1}{\lambda^2}$

$$\text{Median}(m) \triangleq P(X \leq m) = 0.5 \Rightarrow \int_{-\infty}^{m} f_X(x)\,dx = 0.5$$

$$\Rightarrow \int_0^m 0.2\, e^{-0.2x}\,dx = 0.5$$

$$\Rightarrow -(e^{-0.2x})\Big|_0^m = -e^{-0.2m} + 1 = 0.5$$

$$\Rightarrow e^{-0.2m} = 0.5 \Rightarrow -0.2m = \ln 0.5 \Rightarrow m = \frac{\ln 0.5}{-0.2} = 3.47$$

Problem 5:

$$\begin{cases} \int_0^1 (ax+bx^2)\,dx = 1 \\ \int_0^1 x(ax+bx^2)\,dx = 0.6 \end{cases} \Rightarrow \begin{cases} \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]\Big|_0^1 = 1 \\ \left[\frac{ax^3}{3} + \frac{bx^4}{4}\right]\Big|_0^1 = 0.6 \end{cases} \Rightarrow \begin{cases} \frac{a}{2} + \frac{b}{3} = 1 \\ \frac{a}{3} + \frac{b}{4} = 0.6 \end{cases}$$

$$\Rightarrow \begin{cases} 3a + 2b = 6 \\ 4a + 3b = 7.2 \end{cases} \begin{matrix} \times 3 \\ \Rightarrow \\ \times 2 \end{matrix} \begin{cases} 9a + 6b = 18 \\ 8a + 6b = 14.4 \end{cases} \Rightarrow \begin{matrix} a = 18 - 14.4 = 3.6 \\ b = -2.4 \end{matrix}$$

a) $P(X < \frac{1}{2}) = \int_0^{\frac{1}{2}} (3.6x - 2.4x^2)\,dx = \left(3.6\frac{x^2}{2} - 2.4\frac{x^3}{3}\right)\Big|_0^{\frac{1}{2}} = 0.35$

b) $\sigma^2 = \int_0^1 x^2 f(x)\,dx - \mu^2 = \left\{3.6\frac{x^4}{4} - 2.4\frac{x^5}{5}\right\}\Big|_0^1 - 0.6^2 \Rightarrow \sigma^2 = 0.06$

$\mu = E\{X\}$ $\qquad$ $var(X)$

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Problem 6:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow c\int_0^1 x^4\,dx = c\,\frac{x^5}{5}\Big|_0^1 = 1 \Rightarrow c = 5$$

$$f(x) = \begin{cases} 5x^4 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

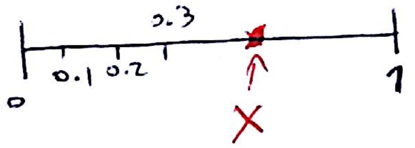
a) $E\{X\} = \int_0^1 x \cdot 5x^4\,dx = 5\,\frac{x^6}{6}\Big|_0^1 = \frac{5}{6}$

b) $var(X) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \mu^2 = \int_0^1 x^2(5x^4)\,dx - (E\{X\})^2 = 5\,\frac{x^7}{7}\Big|_0^1 - \left(\frac{5}{6}\right)^2$

$var(X) = 0.0198$

Problem 7:

$X \sim U(0,1)$



You win three rounds

// // two //

// // one round

a) $P(\underset{\substack{\downarrow \\ \text{win in round 1}}}{w_1}) = P(X>0.1) = \int_{0.1}^1 dx = 0.9$

b) $P(w_2|w_1) = P(X>0.2\,|\,X>0.1) = \frac{P(\overbrace{X>0.2 \cap X>0.1}^{X>0.2})}{P(X>0.1)} = \frac{\int_{0.2}^1 dx}{\int_{0.1}^1 dx} = \frac{0.8}{0.9} = \frac{8}{9}$

c) $P(w_3|w_1 w_2) = P(X>0.3\,|\,\underbrace{X>0.2 \cap X>0.1}_{X>0.2}) = \frac{P(X>0.3 \cap X>0.2)}{P(X>0.2)} = \frac{P(X>0.3)}{P(X>0.2)} = \frac{0.7}{0.8}$

d) You must win in three rounds: $P(w_3 \cap w_2 \cap w_1) = P(w_3 \underbrace{w_2 w_1}_{w_{21}}) = P(w_3 w_{21})$

New event

$P(w_3 w_{21}) = P(w_3 | w_{21})\, P(w_{21}) = P(w_3 | w_2 w_1)\, P(w_2 w_1) = P(w_3 | w_2 w_1)\, P(w_2 | w_1)\, P(w_1)$

Note: $\boxed{P(ABC) = P(A|BC)\, P(B|C)\, P(C)}$

$P(w_3 w_2 w_1) = \text{part c} \times \text{part b} \times \text{part a} = \frac{7}{\cancel{8}} \cdot \frac{\cancel{8}}{\cancel{9}} \cdot \frac{\cancel{9}}{10} = 0.7$

In this easy case you could also find result as follows:

$P(w_3 w_2 w_1) = P(X > 0.3 \cap X > 0.2 \cap X > 0.1) = P(X > 0.3) = 0.7$

~~Because~~ Since the rule of the game is easy. But in many cases finding $P(w_3 w_2 w_1)$ is much harder than $P(w_3 | w_2 w_1)$, $P(w_2 | w_1)$, and $P(w_1)$

Problem 8:

$X_i \sim \exp(\lambda_i)$

Lifetime of type i battry

Lifetime of randomly chosen battry

$P(X > t + s \mid X > t) = ☆$

If there was only one type $\Rightarrow$ ☆ $= P(X > s)$ exponential r.v is memoryless

we have two types

Type 1 $\rightarrow$ X_1 is life time

Type 2 $\rightarrow$ X_2 is lifetime

☆ whenever you can transfer the problem into branches you must use conditional probability

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$$P(X>t+s \mid X>t) = \frac{P(X>t+s \cap X>t)}{P(X>t)} = \frac{P(X>t+s)}{P(X>t)} = ?$$

$$= \frac{P(X>t+s \mid \text{Type 1 is chosen})\, P(T_1) + P(X>t+s \mid T_2)\, P(T_2)}{P(X>t \mid T_1)\, P(T_1) + P(X>t \mid T_2)\, P(T_2)}$$

$$= \frac{P(X_1>t+s)\, P_1 + P(X_2>t+s)\, P_2}{P(X_1>t)\, P_1 + P(X_2>t)\, P_2}$$

we know, $X \sim \exp(\lambda)$

$$P(X>c) = 1 - P(X \leq c) = 1 - [1 - e^{-\lambda c}] = e^{-\lambda c}$$

$$= \frac{e^{-\lambda_1 (t+s)} P_1 + e^{-\lambda_2 (t+s)} P_2}{e^{-\lambda_1 t} P_1 + e^{-\lambda_2 t} P_2}$$

Note: we also can find CDF by using conditioning.

$$P(X \leq c) = P(X \leq c \mid T_1) P(T_1) + P(X \leq c \mid T_2) P(T_2)$$

X: Life time of randomly chosen battry

$$= P(X_1 \leq c) P_1 + P(X_2 \leq c) P_2$$

$$= [1 - e^{-\lambda_1 c}] P_1 + [1 - e^{-\lambda_2 c}] P_2$$

Problem 9: $X \sim \text{bin}(n, \frac{1}{38})$

X: Number of bets you won

$\frac{1}{38}$: Probability of winning

a) $P(X>28) = \sum_{k=28}^{1000} \binom{1000}{k} \left(\frac{1}{38}\right)^k \left(\frac{37}{38}\right)^{1000-k}$ Hard to compute

we can approximate by normal r.v when $n \gg 1$ $(n=1000)$

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$Y \sim N(E\{x\}, var\{x\})$

Normal

$np = \frac{1000}{38} = 26.32$

$np(1-p) = 25.62$

$P(X > 28) \simeq P(Y > 28.5)$

Correction

for converting from discrete to continous

we have table for $\phi(z)$ which is CDF of Standard Normal r.v

$$P\left(\frac{Y - 26.32}{\sqrt{25.62}} > \frac{28.5 - 26.32}{\sqrt{25.62}}\right) = P(\underset{\text{Standard}}{Z} > 0.43) = 1 - P(Z < 0.43)$$

$$= 1 - \phi(0.43) = 1 - 0.6664 = \underline{0.3336}$$

b) $np = 263.16$ $np(1-p) = 256.23$

$P(X > \cancel{280}\ 270) \simeq P(Y > 270.5) = P(Z > 0.45)$

Important note:

Binomial(n, p)

$\begin{cases} n \to \infty \\ P \text{ is not small} \end{cases}$ $\Rightarrow$ only can be approximated by Normal random variable $N(np, np(1-p))$

$\begin{cases} n \to \infty \\ P \to 0 \end{cases}$ $\Rightarrow$ Can be approximated by $\begin{cases} Pois(np) \\ \text{or} \\ N(np, np(1-p)) \end{cases}$

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