Solution. fourth tutorial session:

Q.1) win : W
Lose : L

Based on question: $\begin{cases} P(W) = \frac{18}{38} \\ P(L) = \frac{20}{38} \end{cases}$

① W $\longrightarrow X = 1\$$

② L, W, W $\longrightarrow X = 1\$$

③ L, W, L
④ L, L, W $\Big\} \longrightarrow X = -1\$$

⑤ L, L, L $\longrightarrow X = -3\$$

a) $P(X>0) = P(X=1) = \frac{18}{38} + \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38}$

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b) $E(x) = \sum x P(x)$

$$P(x) = \begin{cases} \frac{18}{38} + \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38} & ; \ x = 1 \\ \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{20}{38} + \frac{20}{38} \cdot \frac{20}{38} \cdot \frac{20}{38} & ; \ x = -1 \\ \frac{20}{38} \cdot \frac{20}{38} \cdot \frac{20}{38} & ; \ x = -3 \end{cases}$$

$E(x) = (1). P(x=1) + (-1) P(x=-1) + (-3) P(x=-1) = \dots$

Q2) a) we must have: $\sum_{K} P(X=K) = 1 \longrightarrow \sum_{K=1}^{4} a.K.2^{-(K-1)} = a + a + \frac{3}{4}a + \frac{a}{2} = 1$

$\Longrightarrow a = 4/13$

$\Longrightarrow P(X=K) = \frac{4}{13} \cdot K \cdot 2^{-(K-1)}$, $K = 1, 2, 3, 4$

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b) $Var(X) = E(X^2) - (E(x))^2$

continued:

$$E(X) = \sum_{K=1}^{4} x\, p(X=x) = (1)p(X=1) + (2)p(X=2) + (3)p(X=3) + (4)p(X=4)$$

$$= (1)\left(\frac{4}{13}\right) + (2)\left(\frac{4}{13}\right) + (3)\left(\frac{3}{13}\right) + (4)\left(\frac{2}{13}\right) = A$$

$$E(X^2) = \sum_{K=1}^{4} x^2 p(X=x) = (1)^2 p(X=1) + (2)^2 p(X=1) + (3)^2 p(X=3) + (4)^2 p(X=4) = B$$

$$\Rightarrow Var(X) = B - A^2 = \ldots.$$

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c) $E(g(x)) = \sum g(x)\, p(g(x))$

$$\Rightarrow E\left(x \log_2 x\right) = \sum_{K=1}^{4} x \log_2 x \; p(X=x) =$$

$$\left(1 \log_2 1\right)\left(\frac{4}{13}\right) + \left(2 \log_2 2\right)\left(\frac{4}{13}\right) + \left(3 \log_2 3\right)\left(\frac{3}{13}\right) + \left(4 \log_2 4\right)\left(\frac{2}{13}\right)$$

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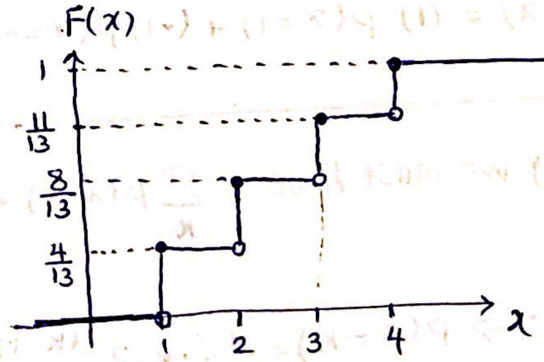
d) $F(1) = P(X \leq 1) = p(X=1)$, $F(2) = P(X \leq 2) = p(X=1) + p(X=2)$

$F(3) = p(X=1) + p(X=2) + p(X=3)$, $F(4) = p(X=1) + p(X=2) + p(X=3) + p(X=4)$

$$F(x) = \begin{cases} 0 & ; \quad x < 1 \\ \frac{4}{13} & ; \quad 1 \leq x < 2 \\ \frac{8}{13} & ; \quad 2 \leq x < 3 \\ \frac{11}{13} & ; \quad 3 \leq x < 4 \\ 1 & ; \quad x \geq 4 \end{cases}$$



f) By definition: $F(z) = P(Z \leq z) = P(2X+3 \leq z) = P\left(X \leq \frac{z-3}{2}\right)$

from the previous part we know that the boundary should be defined as: $\bigcirc < 1$, $1 \leq \bigcirc < 2$, $2 \leq \bigcirc < 3$, $3 \leq \bigcirc < 4$, $\bigcirc \geq 4$

therefore:

$$F(z) = \begin{cases} 0 & ; \quad \frac{z-3}{2} < 1 \longrightarrow z < 5 \\ \frac{4}{13} & ; \quad 1 \leq \frac{z-3}{2} < 2 \longrightarrow 5 \leq z < 7 \\ 8/13 & \quad 2 \leq \frac{z-3}{2} < 3 \longrightarrow 7 \leq z < 9 \\ 11/13 & \quad 3 \leq \frac{z-3}{2} < 4 \longrightarrow 9 \leq z < 11 \\ 1 & \quad \frac{z-3}{2} \geq 4 \longrightarrow z \geq 11 \end{cases}$$

Q3)

a) X = The amount you win. Therefore: $\begin{cases} x = 1.10\ \$ \\ x = -1\ \$ \end{cases}$

$$P(X = 1.10) = \frac{\binom{5}{2} + \binom{7}{2}}{\binom{12}{2}} = \frac{31}{66}$$

$$P(X = -1) = \frac{\binom{5}{1}\binom{7}{1}}{\binom{12}{2}} = \frac{35}{66}$$

$$E(x) = (1.10)\left(\frac{31}{66}\right) + (-1)\left(\frac{35}{66}\right)$$

$$Var(x) = E(x^2) - (E(x))^2 = \left[(1.10)^2\left(\frac{31}{66}\right) + (-1)^2\left(\frac{35}{66}\right)\right] - \left[(1.10)\left(\frac{31}{66}\right) + (-1)\left(\frac{35}{66}\right)\right]^2$$

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Q4. This question was selected to testify the fact that:

when n is large enough.
p is small enough.
np is moderate number

binomial distribution $\approx$ poisson distribution
with parameter of $\lambda = np$.

- using binomial distribution: $p = \binom{10}{0}(0.1)^0(0.9)^{10} + \binom{10}{1}(0.1)^1(0.9)^9 \simeq .07361$

$\lambda = np = (10)(0.1) = 1$.

using poisson: $e^{-1}\frac{(1)^0}{0!} + e^{-1}\frac{(1)^1}{1!} = e^{-1} + e^{-1} \simeq 0.7358$

★ so as you see $0.7361 \simeq 0.7358$.

Q5). $N \longrightarrow$ random variable determining the expected number of games before a win (by A or B)

$p(N=1) = 0$; because at least we need two wins.

$p(N=2) = p^2 + q^2$; where $q = 1-p$. because either A or B succesively should win the game. for $A \longrightarrow p^2$
for $B \longrightarrow q^2$

$p(N=3) = 2pqp + 2qpq$; since, if A wins, we may have:
$= 2p^2q + 2q^2p$

(A, B, A) or (B, A, A)
$p, q, p \; + \; q, p, p \; = 2pqp$

and if B wins, we have:

(B, A, B) or (A, B, B)
$q, p, q \; + \; p, q, q \; = 2qpq$

$P(N=4)=0$; It is obvious that game finished before $N=4$.

$$\Rightarrow E(N) = (2)P(N=2) + (3)P(N=3) = 2(p^2+q^2) + 3(2p^2q + 2q^2p) =$$

$$2+2p-2p^2$$

$$\frac{\partial E(N)}{\partial p} = 2-4p = 0 \Rightarrow p = \frac{1}{2}.$$

Q6) we have 16 effective items

4 defective items.

N: $\longrightarrow$ number of defective items.

$$P(N=0) = \frac{\binom{16}{3}\binom{4}{0}}{\binom{20}{3}} = 0.491$$

$$P(N=1) = \frac{\binom{16}{2}\binom{4}{1}}{\binom{20}{3}} = 0.421$$

$$P(N=2) = \frac{\binom{16}{1}\binom{4}{2}}{\binom{20}{3}} = 0.084$$

$$P(N=3) = \frac{\binom{16}{0}\binom{4}{3}}{\binom{20}{3}} = 0.0035$$

So: $E(N) = (0)P(N=0) + (1)P(N=1) + (2)P(N=2) + (3)P(N=3)$

Q7) This question is important because the interval 12:00_12:05 is different from 2 minutes given in the question.

in the Continous case we have $e^{-(\lambda t)} \frac{(\lambda t)^k}{k!}$ for poisson distribu

where λ is rate of sth (like arrivals) per unit.

in this question, every 2^{min} we have 1 arrival $\Rightarrow$

we have $\frac{1}{2}$ arrival per minutes.

for 12:00 – 12:05 $\longrightarrow$ the interval is <u>5</u> minutes.

So $\lambda t = \frac{1}{2} \times 5 = \frac{5}{2} = 2.5$

$$\Rightarrow P(N=0) = e^{-2.5} = 0.0821$$

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b) $P(N \geqslant 4) = 1 - P(N \leqslant 3) = 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!}\right]$

$\underset{\lambda=2.5}{=\!=\!=} 0.242.$