Session 3
(Tutorial Notes)

Topic: Conditional Probability

1. Why we need to know how to compute "conditional Probability"?

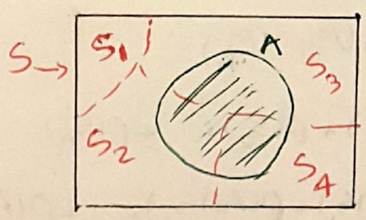
First: In nature events are dependant and because of this, occurrence of an event can change probability of another event. For modelling this new probability, we need to compute conditional probability:

A & B are dependant $\Rightarrow \begin{cases} P(A|B) \neq P(A) \\ P(B|A) \neq P(B) \end{cases}$

$$\begin{cases} P(A|B) = \dfrac{P(A\cap B)}{P(B)} \\ P(B|A) = \dfrac{P(A\cap B)}{P(A)} \end{cases}$$

Trap 1: If you have two events: A & B, and you know: 1. A & B are dependant 2. Event B has happened. In this case you can not use $P(A)$ as probability of event A anymore!

Second: Sometimes we only have one event and we want to compute probability of that event, but ~~see~~ for some reasons, which we will see in this session, we can not find probability in regular way (Counting). In this case we need to divide the ~~event~~ sample space into some "Mutually Exclusive" sub-events:



$$P(A) = P((S_1\cap A)\cup(S_2\cap A)\cup\cdots(S_n\cap A))$$

Mutually Exclusive $= P(S_1\cap A) + \cdots + P(S_n\cap A)$

$$= P(A|S_1)P(S_1) + \cdots + P(A|S_n)P(S_n)$$

$$= \sum_{i=1}^{n} P(A|S_i)P(S_i)$$

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Trap 2: If chosen Sub-events are not "Mutually Exclusive", we can not write union of them as summation of probability of each one.

Example: A = Two heads when coin is flipped two times

S_1 = First flip is head $\quad S_3 = HH$ (first, second)

S_2 = " " " Tail $\quad S_4 = TH \quad S_5 = HT \quad S_6 = TT$

$S_1 \cup S_2 \cup \dots \cup S_6 = S$ ✓ First condition

But: $S_1 \cap S_3 \neq \phi$ ✗

$$P(A) = \frac{1}{4} \neq \underbrace{P(A|S_1)}_{\frac{1}{2}} \times \underbrace{P(S_1)}_{\frac{1}{2}} + \underbrace{P(A|S_2)}_{0} \times \underbrace{P(S_2)}_{\frac{1}{2}} + \underbrace{P(A|S_3)}_{1} \times \underbrace{P(S_3)}_{\frac{1}{4}} + \underbrace{P(A|S_4)}_{0} \times \underbrace{P(S_4)}_{\frac{1}{2}}$$

$$+ \underbrace{P(A|S_5)}_{0} \times \underbrace{P(S_5)}_{\frac{1}{2}} + \underbrace{P(A|S_6)}_{0} \times \underbrace{P(S_6)}_{\frac{1}{2}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

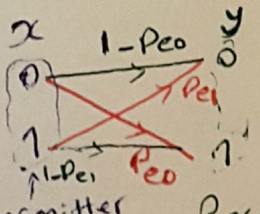
Trap 3: Union of all sub-events should be whole sample space (S).

2. Different types of problems:

a) We have $P(B|A)$, but we want: $P(A|B)$ or $P(A)$ or $P(B)$

Problem 1: We have a communication system in which we send one bit $\{0,1\}$, through a channel, with $\{P_0, P_1\}$. In channel, with probability P_{e0}, $0 \to 1$ and P_{e1}, $1 \to 0$

find:



P_0: Probability of sending 0 $\quad P(x=0)$

P_1: " " " 1 $\quad P(x=1)$

P_{e0}: 0 is sent $\longrightarrow$ Probability of receiving 1

$$P(y=1 | x=0) = 1 - P(y=0 | x=0)$$

~~a) $P(y=1|x=0)$ P_{e0}~~

P_{e1}: $P(y=0 | x=1) = 1 - P(y=1 | x=1)$

$= \Sigma P$

a) Find probability of receiving $\underline{0}$:

$$P(y=0) = P(y=0|x=0)\,P(x=0) + P(y=0|x=1)\,P(x=1)$$

Mutually Exclusive

$$P(y=0) = [1-P_{e_0}]\,P_0 + P_{e_1} P_1$$

b) If $\underline{1}$ is received, find the probability that $\underline{0}$ has been sent :

$y=1$ $\quad$ $x=0$

$$P(x=0 \mid y=1)$$

$$P(x=0|y=1)\,\frac{P(y=1)}{P(y=1)} = \frac{P(x=0 \cap y=1)}{P(y=1)} = \frac{P(y=1|x=0)\,P(x=0)}{P(y=1)}$$

we know values of these two

we do not know value of this

$$\Rightarrow \quad P(x=0|y=1) = \frac{P_{e_0} P_0}{P(y=1|x=0)P(x=0) + P(y=1|x=1)P(x=1)}$$

$$= \frac{P_{e_0} P_0}{P_{e_0} P_0 + [1-P_{e_1}]\,P_1}$$

Trap4: Be careful when you want transfer question into mathematical expression

$\underbrace{\text{If statement}}_{A}$, $\overbrace{\text{Result}}^{B}$

$\Downarrow$ Translation

$$P(\underset{B}{\text{Result}} \mid A)$$

Problem 2: (Winter 2012)
Midterm

A student has asked his supervisor for a letter of recommendation for an award. He estimates that there is a 90% chance that he will get the award if he receives a strong recommendation, a 50% chance if he receives a moderately good recommendation, and a 10% chance if he receives a weak recommendation. He further estimates that the probabilities that the recommendation will be strong, moderate, and weak are 0.7, 0.2, and 0.1, respectively.

(a) (5 points) How certain is he that he will receive the award?

(b) (10 points) Given that he does receive the award, how likely should he feel that he received a strong recommendation?

(c) (10 points) Given that he does not receive the award, how likely should he feel that he received a weak recommendation?

Solution:

A: The event that he receives the award

If recommendation is Strong $\longrightarrow$ Chance of getting award $= 0.9$

$\Downarrow$

$$\begin{cases} P(A|S) = 0.9 \\ P(A|G) = 0.5 \\ P(A|W) = 0.1 \end{cases}$$

Good

Also $\begin{cases} P(S) = 0.7 \\ P(G) = 0.2 \\ P(W) = 0.1 \end{cases}$

a) $P(A) = ?$

$$P(A) = P(A|S)P(S) + P(A|G)P(G) + P(A|W)P(W)$$

$$= (0.9)(0.7) + (0.5)(0.2) + (0.1)(0.1) = 0.74$$

b) $P(\text{strong}|A) = ?$

Bayes' formula,

$$P(S|A) = \frac{P(A|S)P(S)}{P(A)} = \frac{(0.9)(0.7)}{0.74} = \frac{63}{74}$$

$\leftarrow$ part a

c) $P(\text{weak}|A^c) = ?$

$$P(A^c|W) = 1 - P(A|W) = 1 - 0.1 = 0.9$$

$$P(A^c) = 1 - P(A) = 1 - 0.74 = 0.26$$

$$P(W|A^c) = \frac{P(A^c|W)P(W)}{P(A^c)}$$

b) We want to find probability of an event, but desired outcomes can not be counted and experiment restarts after some trials.

Problem 3:

> A and B roll a pair of dice in turn, with A rolling first. A's objective is to obtain a sum of 6, and B's is to obtain a sum of 7. The game ends when either player reaches his or her objective, and that player is declared the winner.
> (a) Find the probability that A is the winner.

Solution:

As you can see in this problem after two rollings problem restarts.

We can not use regular method for finding this probability because the End point is not known. If we want to use regular methods:

P(A win) = P(A wins in first trial) + P(A wins in third trial & B does not win in second trial) + ...

A wins in first trial

A wins in third trial

P(A win) = P(Sum of 6 in first trial) + P(First rolling sum is not 6 ∩ Second rolling sum is not 7 ∩ Third rolling sum is 6) + ...

A B A

P(a sum of 7)

$$P(A_{win}) = \frac{5}{36} + \left(1-\frac{5}{36}\right)\left(1-\frac{1}{6}\right)\left(\frac{5}{36}\right) + \left(\frac{31}{36}\right)^2\left(\frac{5}{6}\right)^2\frac{5}{36} + \cdots$$

P(sum of 6) P(A sum of 6) P(a sum of 6)

$$P(A) = \frac{5}{36}\underbrace{\left(1+\left(\frac{31}{36}\right)\left(\frac{5}{6}\right)+\cdots\right)}_{\text{Geometric series}} = \frac{5}{36}\,\frac{1}{1-\frac{31\times 5}{36\times 6}} = \frac{5}{\cancel{36}}\times\frac{\cancel{36}\times 6}{61} = \boxed{\frac{30}{61}}$$

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Problem 3 cont'd:

But by using conditioning we can find result much easier:

Conditioning on first rolling:

$$P(A\text{ wins}) = \underbrace{P(A\text{ wins} \mid \text{In first trial sum is } 6)}_{=1}\, \underbrace{P(\text{sum is } 6)}_{=\frac{5}{36}} +$$

$$\underbrace{P(A\text{ wins} \mid \text{ " " " is not } 6)}_{?}\, \underbrace{P(\text{ " " not } 6)}_{=\frac{31}{36}}$$

Conditioning on second trial:

$$P(A\text{ wins} \mid \text{sum is not } 6 \text{ in first trial}) = P(A\text{ wins} \mid \text{sum is not } 6, \text{ sum is } 7)\, P(\text{sum is } 7) +$$

Second trial

$$P(A\text{ wins} \mid \underset{\text{First}}{\text{not } 6}, \underset{\text{Second}}{\text{not } 7})\, P(\text{sum is not } 7)$$

$$= 0 \times \frac{1}{6} + P(A\text{ wins}) \times \frac{5}{6}$$

Since in this situation B wins

Game restarts

$$P(A\text{ wins}) = 1 \times \frac{5}{36} + \left(0 \times \frac{1}{6} + P(A\text{ wins}) \times \frac{5}{6}\right) \times \frac{31}{36}$$

$$P(A) = \frac{5}{36} + \frac{5}{6} \times \frac{31}{36} \times P(A)$$

$$\boxed{P(A) = \frac{30}{61}}$$

Problem 4:

Two players take turns shooting at a target, with each shot by player i hitting th target with probability p_i, $i = 1, 2$. $\rightarrow \rho_i$

Suppose that the shooting ends when the target has been hit twic

Let m_i d~~enote the mean number of shots needed for the first hit when player i shoo first, $i = 1, 2$. Also,~~ let P_i, $i = 1, 2$, denote the probability that the first hit is b player 1, when player i shoots first. $\rightarrow P_i$

(a) Find m_1 and m_2. Ch 7

(b) Find P_1 and P_2.

For the remainder of the problem, assume that player 1 shoots first.

(c) Find the probability that the final hit was by 1.

(d) Find the probability that both hits were by 1.

(e) Find the probability that both hits were by 2.

(f) Find the mean number of shots taken. Ch 7

Solution:

As you can see the competition restarts after two times shooting and end point is not fixed, so conditioning can help in finding result:

b) P_i: First hit is by player 1, when player i shoots first.

P_1 = P(Player 1's first shot hit | First shot hit) p(First shot hit) +
P(" " " " | " " did not hit) p(" " did not hit)

Player 1 starts

$$\begin{cases} P_1 = 1 \times \rho_1 + P_2 \times \underbrace{q_1}_{=1-\rho_1} \\ P_2 = 0 \times \rho_2 + P_1 \times q_2 \end{cases} \Rightarrow P_1 = \frac{\rho_1}{1 - q_1 q_2} \qquad P_2 = \frac{\rho_1 q_2}{1 - q_1 q_2}$$

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c) $p(L_i) \triangleq$ Probability that the last hit is by player 1, when player i shoots first.

Conditioning on the outcome of the first shot gives:

$$p(L_1) = p_1 \underline{p}_2 + q_1 p(L_2)$$

- L_1: Player 1 starts the game
- p_1: First shot hits
- q_1: First shot does not hit
- $\underline{p}_2$: we need one other shot to hit but player 2 now starts

1	2	
✓		We only need one other hit so we can assume 2 starts game and first hit should be by 1
⊗		we need two hits so game restarts and we want {probability that the last hit is by player 1, when player 2 starts shooting game}

($\underline{p}_2$: Probability that the first hit is by 1, when player 2 shoots first)

$$L_2 = p_2 \underline{p}_1 + q_2 L_1 \Rightarrow L_1 = \frac{p_1 \underline{p}_2 + q_1 p_2 \underline{p}_1}{1 - q_1 q_2}$$

(d & e) $B_i \triangleq$ Both hits were by i (1 starts the game)

Conditioning on first two shots:

	1	2
S_1:	✓	✓
S_2:	✓	⊗
S_3:	⊗	✓
S_4:	⊗	⊗

$P(B_1 | S_1) = 0$ Because we have two hits so game is finished but both were not by 1

$P(B_1 | S_2) = \underline{p}_1$ we only need one other hit and that should be by player 1 (First hit by 1 when 1 starts)

$P(B_1 | S_3) = 0$ one hit was by 2 so both can not be by 1

$P(B_1 | S_4) = P(B_1)$ Game restarts and there have not been any hit yet.

$$P(B_1) = \underline{p}_1 \underbrace{p_1 q_2}_{P(S_2)} + P(B_1) \underbrace{q_1 q_2}_{P(S_4)} \Rightarrow P(B_1) = \frac{p_1 q_2 \underline{p}_1}{1 - q_1 q_2}$$

why? →

$$P(B_2) = (1 - \underline{p}_1) q_1 p_2 + P(B_2) q_1 q_2 \Rightarrow P(B_2) = \frac{q_1 p_2 (1 - \underline{p}_1)}{1 - q_1 q_2}$$

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Problem 5: A, B, and C are evenly matched tennis players. Initially A and B play a set, and the winner then plays C. This continues, with the winner always playing the waiting player, until one of the players has won two sets in a row. That player is then declared the overall winner. Find the probability that A is the overall winner.

$P(A)$: The probability that $\underline{A}$ wins.

$P(B)$: 〃 〃 〃 $\underline{B}$ 〃

$P(C)$: 〃 〃 〃 $\underline{C}$ 〃 $= P(R)$: The probability that the waiting player wins whole game

A&B Starts. Conditioning on first match: (w: A wins L: A loses)

$$P(A) = \frac{1}{2}P(A|w) + \frac{1}{2}P(A|L)$$

A vs B (L, w), C waiting ⇒ B vs C, A waiting ⇒ B vs C (w, L) ; B vs C (L, w)

A is waiting person

$$P(A|L) = P(A|L,(w\ L)) \times \frac{1}{2} + P(A|L,(L\ w)) \times \frac{1}{2} = 0 \times \frac{1}{2} + P(R) \times \frac{1}{2} = \boxed{\frac{1}{2}P(C)} \star$$

Game ends and $\underline{B}$ is winner

$P(A|w)$: A vs B (w, L) ⇒ A vs C (w, L) (Game ends and $\underline{A}$ is winner) ; A vs C (L, w) ⇒ C vs B (w, L) (Game ends and $\underline{C}$ is winner) ; C vs B (L, w)

$$P(A|w) = \underbrace{P(A|w,(w,L))}_{1} \times \frac{1}{2} + \underbrace{P(A|w,(L,w),(w,L))}_{0\ (\underline{C}\text{ wins})} \frac{1}{2} \times \frac{1}{2} + \underbrace{P(A|w,(L,w),(L,w))}_{P(R)=P(C)} \times \frac{1}{4}$$

$$P(A|w) = \frac{1}{2} + \frac{1}{4}P(C)$$

$$P(A) = \frac{1}{4} + \frac{1}{8}P(C) + \frac{1}{4}P(C) = \frac{1}{4} + \frac{3}{8}P(C)$$

Also:

$$P(C) = \frac{1}{2}P(C|\overset{AB}{(W,L)}) + \frac{1}{2}P(C|\overset{AB}{(L,W)}) = P(C|\overset{AB}{(W,L)})$$

$\oslash$

$$\oslash = \frac{\underbrace{P(C|\overset{AB}{(W,L)}\overset{AC}{(W,L)})}_{=0}}{2} + \frac{P(C|\overset{AB}{(W,L)},\overset{AC}{(L,W)})}{2} \quad \triangle$$

(Since A wins the game)

$$\triangle = \underbrace{P(C|\overset{AB}{(W,L)}\overset{AC}{(L,W)}\overset{CB}{(W,L)})}_{=1} \times \frac{1}{2} + \underbrace{P(C|\overset{AB}{(W,L)}\overset{AC}{(L,W)}\overset{CB}{(L,W)})}_{=P(A|L)} \times \frac{1}{2}$$

1 (C wins)

$P(A|L) = \frac{1}{2}P(R) = \frac{1}{2}P(C)$ $\circledast$

$$P(C) = 0 + \frac{\triangle}{2} = \frac{\frac{1}{2} + \frac{P(C)}{4}}{2} \Rightarrow P(C) = \frac{2}{7} \qquad P(A) = P(B) = \frac{5}{14}$$

Note: $\circledast$ is because now we have:

$\underset{L}{C}$ vs $\underset{W}{B}$

A waiting

The clever part is that we can change name of players

$\underset{L}{A}$ vs $\underset{W}{B}$

C waiting

It is similar to case that A & B starts the game and we want probability that A wins the game conditioned on the fact that he lost first game $P(A|L)$

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:) Conditioning helps us to find a recursive formula for finding probability

Problem 6:

Independent trials that result in a success with probability p and a failure with probability $1 - p$ are called *Bernoulli trials*. Let P_n denote the probability that n Bernoulli trials result in an even number of successes (0 being considered an even number). Show that

$$P_n = p(1 - P_{n-1}) + (1 - p)P_{n-1} \quad n \geq 1$$

and use this formula to prove (by induction) that

$$P_n = \frac{1 + (1 - 2p)^n}{2}$$

In this problem we should find a recursive formula:

Conditioning on result of first trial:

$$P_n = \underbrace{P_n(\text{Even}|S)}_{\Downarrow} \underset{P(S)}{p} + \underbrace{P_n(\text{Even}|F)}_{\Downarrow} \underset{P(F)}{(1-p)} = p(1 - P_{n-1}) + (1-p)P_{n-1}$$

If the first trial is a success then we have $n-1$ trials to go and to obtain an even total successes the number of successes in this $n-1$ trials to be odd

$(1 - P_{n-1})$

Even number of successes in $n-1$ trials

we need even number of successes in $n-1$ trials

(P_{n-1})

Proof by induction:

1) $$\begin{cases} P_1 = p(1-\underbrace{P_0}_{1}) + (1-p)P_0 = 1-p = q \\ \text{Formula: } P_1 = \dfrac{1+(1-2p)^1}{2} = 1-p \end{cases}$$ ✓

2) Assume that it holds for $n-1$

3) Try to find P_n:

$$P_n = p(1-P_{n-1}) + (1-p)P_{n-1} \overset{\text{Assumption}}{=} p + (1-2p)\left(\frac{1+(1-2p)^{n-1}}{2}\right)$$

$$= p + \frac{1-2p}{2} + \frac{(1-2p)^n}{2} = \frac{1}{2} + \frac{(1-2p)^n}{2} \quad (\star)$$

From formula: $P_n = \dfrac{1+(1-2p)^n}{2} = (\star)$ ✓ $\boxed{\text{proved}}$

Another way (Solving difference equation directly):

$$P_n = p(1-P_{n-1}) + (1-p)P_{n-1}$$

Homogenous answer: $P_n + pP_{n-1} + pP_{n-1} - P_{n-1} = 0$

$$z^n + pz^{n-1} + pz^{n-1} - z^{n-1} = 0 \Rightarrow z + (2p-1) = 0 \Rightarrow \boxed{z = 1-2p}$$

$$P_n = C \Rightarrow C = p(1-C) + (1-p)C \Rightarrow C = \frac{1}{2}$$

$$\begin{cases} P_n = C + C_1 z^n \\ C = \frac{1}{2} \\ P_0 = 1 \end{cases} \Rightarrow 1 = C + C_1 \Rightarrow C_1 = \frac{1}{2} \Rightarrow P_n = \frac{1}{2} + \frac{1}{2}(1-2p)^n$$

Problem 7 **(The Ballot Problem)** In an election, candidate A receives n votes, and candidate B receives m votes where $n > m$. Assuming that all orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n-m)/(n+m)$.

$P_{n,m} \triangleq$ Probability that A is always ahead in the count of votes, If A receives n votes and B receives m votes.

We try to find a recursive formula by conditioning on last vote:

why?

$$P_{n,m} = P_{n-1,m} \underbrace{\frac{n}{n+m}}_{P(\text{Last vote is for } A)} + P_{n,m-1} \underbrace{\frac{m}{n+m}}_{P(\text{Last vote is for } B)}$$

Important: why it would not be helpful if we condition on first vote?

Because $P_{n-1,m}$ & $P_{n,m-1}$ would not appear in formula

A . . . .

First vote

$P(A \text{ ahead} \mid \text{first vote is for } A) > P_{n-1,m}$

$\Downarrow$

First vote A A B . . . ✓ But it has not been counted in $P_{n-1,m}$

By induction:

1) $n=2, m=1$ AAB $P_{2,1} = \dfrac{1}{\dfrac{3!}{\underbrace{2!}_{A's}\ \underbrace{1!}_{B}}} = \dfrac{1}{3} = \dfrac{2-1}{2+1}$ ✓

2) Assume that it is correct for $(n-1,m)$ & $(n,m-1)$

3) $$P_{n,m} = \frac{n-1-m}{n+m-1} \, \frac{n}{n+m} + \frac{n-m+1}{n+m-1} \, \frac{m}{n+m} = \frac{n^2-n-nm+nm-m^2+m}{(n+m)(n+m-1)}$$

$$= \frac{n^2-m^2+m-n}{(n+m)(n+m-1)} = \frac{(n-m)\cancel{(n+m-1)}}{(n+m)\cancel{(n+m-1)}} = \frac{n-m}{n+m}$$ ✓

Problem 8:

Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0?

Conditioning on first gambling:

$P_i \triangleq$ probability that starting by i units, the gambler's fortune will reach N before 0.

$P_i = p P_{i+1} + q P_{i-1}$ (why?)

i units

wins p — Now the person has $i+1$ units

Loses q — 〃 〃 〃 〃 $i-1$ 〃

First gambling

$\Downarrow$

The game restarts. now we can think the person at first has $i+1$ units (First branch)

〃 〃 〃 〃 〃 $i-1$ 〃 (Second 〃)

or 〃 〃 〃

So we have:

$$P_i = p P_{i+1} + q P_{i-1}$$

$$\underbrace{(p+q)}_{1} P_i = p P_{i+1} + q P_{i-1}$$

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1})$$

$i=1 \quad \cancel{P_2} - P_1 = \frac{q}{p}(P_1 - \overset{=0}{P_0}) = \frac{q}{p}P_1$

$P_3 - \cancel{P_2} = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$

$\vdots \; P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$

Adding all $i-1$ equations

$P_N - P_{N-1} = \left(\frac{q}{p}\right)^{N-1} P_1$

$$\Rightarrow P_i - P_1 = P_1 \underbrace{\left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]}_{\text{Geometric Series}}$$

$\hookrightarrow (1 + G + G^2 + \cdots + G^N) = \frac{1 - G^{N+1}}{1 - G}$

$$P_i = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \frac{q}{p}} P_1 & \frac{q}{p} \neq 1 \\ i P_1 & \frac{q}{p} = 1 \end{cases}$$

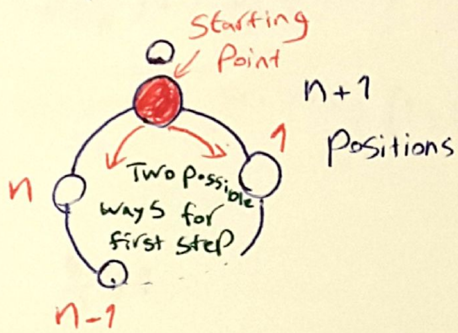
We know: $P_N = 1 \Rightarrow P_1 = \begin{cases} \dfrac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N} & \frac{q}{p} \neq 1 \\ \dfrac{1}{N} & \frac{q}{p} = 1 \end{cases}$

$$\boxed{P_i = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N} & \frac{q}{p} \neq 1 \\ \dfrac{i}{N} & \frac{q}{p} = 1 \end{cases}}$$

Problem 9:

A particle moves among $n+1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $q = 1 - p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the probability that all states have been visited by time T.

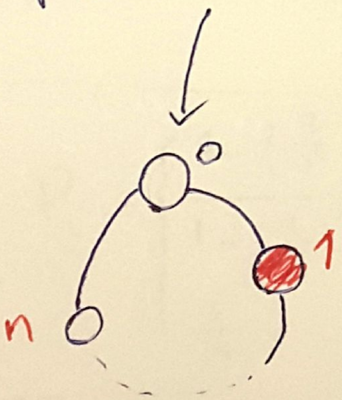
We can think about each step as a trial.



Conditioning on first step:

A: Event that all states have been visited by time T.

$P(A) = P(A \mid \text{First step clockwise})\,p + P(A \mid \text{First step Counter clockwise})\,q$



So at each step (trial) Position (Pos) can be

Pos+1, by probability p, or Pos−1, by Probability $1-p=q$.

We want to find $\Rightarrow$ $\boxed{P(\text{Pos}=n \text{ before Pos}=0)}$

So it is same as gambler's ruin problem in which gambler starts with 1 unit (First position)

$$P(A) = \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^n} p + q \frac{1 - \frac{p}{q}}{1 - \left(\frac{p}{q}\right)^n}$$