Session 12, Solutions

Q1.

a)

$$\mu_x = 0 \Longrightarrow \mu_y = \mu_x \int_0^\infty h(t)dt = 0$$

b)

$$S_x(f) = \mathcal{F}[\delta(\tau)] = 1$$

$$H(f) = \mathcal{F}[h(t)] = \frac{1}{1 + j2\pi f} \Longrightarrow |H(f)|^2 = \frac{1}{1 + (2\pi f)^2}$$

Thus,

$$S_Y(f) = |H(f)|^2 S_x(f) = \frac{1}{1 + 4\pi^2 f^2}$$

c)

$$R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = \mathcal{F}^{-1}[\frac{2}{1 + 4\pi^2 f^2}.1/2] = 1/2e^{-|\tau|}$$

d) Yes, $Y(t)$ is a WSS process, since $\mu_y = 0$ and $R_Y(\tau)$ only depends on τ.

Q2.

(6) 6.1. $E[X(t)] = E[A(t)\cos(\omega t+\theta)]$

$$= \int_0^{2\pi} \frac{1}{2\pi} A(t)\cos(\omega t+\theta)\,d\theta = \frac{A(t)}{2\pi}\underbrace{\sin(\omega t+\theta)\Big|_0^{2\pi}}_{0} = 0$$

6.2. $R_X(t,\tau) = E[X(t)X(t+\tau)] = E[A(t)\cos(\omega t+\theta)A(t+\tau)\cos(\omega t+\omega\tau+\theta)]$

$$= \frac{1}{2\pi}A(t)A(t+\tau)\int_0^{2\pi}\underbrace{\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)}_{\frac{1}{2}[\cos(2\omega t+\omega\tau+\theta)+\cos(\omega\tau)]}\,d\theta$$

$$= \frac{1}{2\pi}A(t)A(t+\tau)\left[\frac{1}{2}\underbrace{\sin(2\omega t+\omega\tau+\theta)\Big|_0^{2\pi}}_{0} + \underbrace{\frac{\theta}{2}\cos(\omega\tau)\Big|_0^{2\pi}}_{\frac{2\pi}{2}\cos(\omega\tau)}\right] = \frac{1}{2}A(t)A(t+\tau)\cos(\omega\tau)$$

6.3. $X(t)$ is not WSS since $R_X(t,\tau)$ is not independent of 't'

6.4. $A(t)=A \Rightarrow R_X(t,\tau) = \frac{1}{2}A^2\cos(\omega\tau) \Rightarrow S_X(f) = \text{F.T.}\{\frac{1}{2}A^2\cos(\omega\tau)\}$

F.T. $\{\cos(2\pi f_0\tau)\} = \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$ $(2\pi f_0 \triangleq \omega)$ define

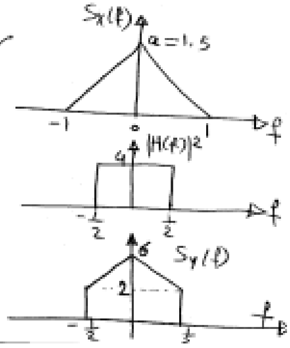
$$S_X(f) = \frac{1}{4}A^2\left[\delta\left(f-\frac{\omega}{2\pi}\right)+\delta\left(f+\frac{\omega}{2\pi}\right)\right]$$

6.5. $R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df \to a=1.5$

$R_X(0) = E[X^2_{(t)}] = 1.5$

$S_Y(f) = |H(f)|^2 S_X(f)$

$R_Y(\tau) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_Y(f)e^{j2\pi f\tau}\,df = \ldots$ You could either take this integral or use table



Q3)

(5) $P\{N(t)=k\} = \begin{cases} \frac{(\lambda t)^k}{k!} e^{-\lambda t} & k=0,1,2,\ldots \\ 0 & \text{O.W.} \end{cases}$

5.1. $P\{N(10)=5\} = \frac{(0.1\times 10)^5}{5!} e^{-0.1\times 10} = \boxed{\frac{1}{120\, e}}$

5.2. $P\{N(2)=0\} = \frac{(0.1\times 2)^0}{0!} e^{-0.1\times 2} = \boxed{e^{-0.2}}$

5.3. $T \sim \text{Exponetial } (\lambda=0.1)$ $\begin{cases} E[T] = \frac{1}{\lambda} = \boxed{10} \\ Var[T] = \frac{1}{\lambda^2} = \boxed{100} \end{cases}$

5.4. $P(T<3) = \int_0^3 \lambda e^{-\lambda t}\, dt = -e^{-\lambda t}\Big]_0^3 = \boxed{1-e^{-0.3}}$

5.5. $P\{N(3)=0\} = \boxed{e^{-0.3}}$ Since poisson process is memoryle

Q4)

a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^2\int_0^2 c(x+y)dxdy = 1 \quad \Rightarrow \quad c = \frac{1}{8}$$

b)

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \int_0^2 \frac{1}{8}(x+y)dx \\ &= \frac{1}{4}(1+y) \quad (0 \le y \le 2) \\ f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(1+y)} = \frac{x+y}{2(1+y)} \end{aligned}$$

c)

$$\begin{aligned} P\Big\{X > \frac{1}{3} \,\Big|\, Y = y\Big\} &= \int_{\frac{1}{3}}^{\infty} f_{X|Y}(x|y)dx = \int_{\frac{1}{3}}^{2} \frac{x+y}{2(1+y)}dx \\ &= \frac{1}{2(1+y)}\Big(\frac{1}{2}x^2 + yx\Big)\Big|_{\frac{1}{3}}^{2} \\ &= \frac{5}{36}\cdot\frac{7+6y}{1+y} \end{aligned}$$

d)

$$\begin{aligned} E[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_0^2 x\cdot\frac{x+y}{2(1+y)}dx \\ &= \frac{4+3y}{3(1+y)} \end{aligned}$$