Solutions.
Session 11

Q1. $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{(2m+6)x^m y}{\int_0^x (2m+6)x^m y\,dy} = \dfrac{(2m+6)x^m y}{(2m+6)\frac{x^{m+2}}{2}} =$

$\dfrac{2y}{x^2}$

$$Var(Y^2|X) = E(Y^4|X=x) - (E(Y^2|X=x))^2$$

$$E(Y^4|X=x) = \int_0^x y^4 f_{Y|X}(y|x)dy = \int_0^x y^4 \frac{2y}{x^2}dy = \frac{2y^6}{6x^2}\Big|_0^x = \frac{x^4}{3}$$

$$E(Y^2|X=x) = \int_0^x y^2 f_{Y|X}(y|x)dy = \int_0^x y^2 \frac{2y}{x^2}dy = \frac{2y^4}{4x^2}\Big|_0^x = \frac{x^2}{2}$$

$$\Longrightarrow Var(Y^2|X) = \frac{x^4}{3} - \left(\frac{x^2}{2}\right)^2 = \frac{x^4}{3} - \frac{x^4}{4} = \frac{x^4}{12}.$$

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Q2) $Z \sim N(0,1) \longrightarrow E(Z)=0,\ Var(Z)=1$

$$\rho(Y,Z) = \frac{cov(Y,Z)}{\sqrt{var(Y)var(Z)}} = \frac{E(ZY) - E(Y)\cancelto{0}{E(Z)}}{\sqrt{Var(Y)}} = \frac{E(ZY)}{\sqrt{Var(Y)}}$$

$$E(Y) = E(a+bZ+cZ^2) = a + b\cancelto{0}{E(Z)} + c\cancelto{1}{E(Z^2)} = a+c$$

$$Var(Y) = E((Y-E(Y))^2) = E((bZ+cZ^2-c)^2) =$$

$$E(b^2Z^2 + c^2Z^4 + c^2 - 2bcZ - 2c^2Z^2 + 2bcZ^3) =$$

(*) $$b^2\underset{1}{\cancel{E(Z^2)}} + c^2E(Z^4) + c^2 - 2bc\underset{0}{\cancel{E(Z)}} - 2c^2\underset{1}{\cancel{E(Z^2)}} + 2bcE(Z^3)$$

$= \downarrow$

we should find $E(Z^3)$ & $E(Z^4)$

$$E(Z^3) = \frac{d^3 M_Z(t)}{dt^3}\Big|_{t=0}$$

$$E(Z^4) = \frac{d^4 M_Z(t)}{dt^3}\Big|_{t=0}$$

and we know that for a standard normal r.v, $M_Z(t) = e^{t^2/2}$

$$e^{t^2/2} \xrightarrow{d/dt} te^{t^2/2} \xrightarrow{d^2/dt^2} e^{t^2/2} + t^2 e^{t^2/2} \xrightarrow{d^3/dt^3}$$

$$te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2} \xrightarrow{d^4/dt^4}$$

$$e^{t^2/2} + t^2 e^{t^2/2} + 2e^{t^2/2} + 2t^2 e^{t^2/2} + 3t^2 e^{t^2/2} + t^4 e^{t^2/2}$$

you can easily check that:

$E(Z^3) = 0$

$E(Z^4) = 3$

put them in the $(*)$, we have: $Var(Y) = b^2 + 2c^2$

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$$E(YZ) = E((a+bZ+cZ^2).Z) = E(aZ + bZ^2 + cZ^3) = a\overset{0}{\cancel{E(Z)}} + b\overset{1}{E(Z^2)} + c\underset{0}{E(Z^3)} = b$$

$$\Rightarrow \rho(Y,Z) = \frac{b}{\sqrt{b^2+2c^2}}$$

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Q3) Since $X$ and $Y$ are independent $\Longrightarrow E(XY) = E(X)E(Y)$

$$E(X) = M_X'(t)\Big|_{t=0} = (2e^t)(e^{2e^t-2})\Big|_{t=0} = 2$$

$$E(Y) = M_Y'(t)\Big|_{t=0} = (10)\left(\frac{3}{4}\right)\left(\frac{3}{4}e^t + \frac{1}{4}\right)^9\Big|_{t=0} = 7.5$$

$$\Longrightarrow E(XY) = E(X)E(Y) = (2)(7.5) = 15$$

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Q(4) $E(X_i) = 1$ , $Var(X_i) = 1$

$$\Longrightarrow Y = \frac{\sum x_i - (20)(1)}{(1)\sqrt{20}} = \frac{\sum x_i - 20}{\sqrt{20}} \sim N(0,1)$$

$$P\left\{\sum_{i=1}^{20} x_i > 15\right\} = P\left\{\frac{\sum x_i - 20}{\sqrt{20}} > \frac{15-20}{\sqrt{20}}\right\} = P\{Y > -1.118\} =$$

$$1 - P\{Y \leq -1.118\} \overset{(a)}{\approx} 0.86$$

(a) where it follows from Normal Table.