Q1. we should use convolution in descrete domain.

define $Z = X + Y$

$$P(Z=n) = \sum_{K=-\infty}^{+\infty} P(X_1 = n-K)\, P(X_2 = K)$$

$$= \sum_{K=0}^{+\infty} e^{-\lambda_1} \frac{\lambda_1^{n-K}}{(n-K)!} e^{-\lambda_2} \frac{\lambda_2^{K}}{K!}$$

because $n-K$ should be $\geqslant 0 \Rightarrow K \leqslant n$.

$$\Rightarrow P(Z=n) = \sum_{K=0}^{n} e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^{n-K}\lambda_2^{K}}{(n-K)!\,K!} =$$

$$\frac{e^{-(\lambda_1+\lambda_2)}}{n!} \times \sum_{K=0}^{n} \frac{n!}{(n-K)!\,K!} \lambda_1^{n-K} \lambda_2^{K} =$$

$$\frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n$$

so it is again poisson distribution with parameter $\lambda_1+\lambda_2$.

Q2) let's define $X = X_1 + X_2 + \cdots + X_n$ and $Y = Y_1 + Y_2 + \cdots + Y_n$, where

$$X_i = \begin{cases} 1 & ; \text{ role } i \text{ lands on } 1 \\ 0 & ; \text{ o.w} \end{cases}$$

$$Y_i = \begin{cases} 1 & ; \text{ role } i \text{ lands on } 2 \\ 0 & ; \text{ o.w.} \end{cases}$$

then $cov(X,Y) = cov\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} cov(X_i, Y_j)$

but $cov(X_i, Y_j) = E(X_i Y_j) - E(X_i)E(Y_j)$

note that $E(x_i) = (x_i=1)p(x_i=1) + (x_i=0)p(x_i=0) = 1/6$

Similarly $E(Y_j) = 1/6$

$$E(X_i Y_j) = (X_i=1)(Y_j=1)p(X_i=1, Y_j=1) = \begin{cases} 0 & ; \quad i=j \\ \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} & ; \quad i \neq j \end{cases}$$

therefore, $$cov(x_i, Y_j) = \begin{cases} \frac{-1}{36} & ; \ i=j \\ \frac{1}{36} - \frac{1}{36} = 0 & ; \ i \neq j \end{cases}$$

Therefore, $cov(X,Y) = (n^2 - n)(0) + (n)\left(\frac{-1}{36}\right) = \frac{-n}{36}$

Q3. let's define $Z_1 = X_1 + X_2$ & $Z_2 = X_2 + X_3$. Then $\rho(Z_1, Z_2) = \frac{cov(Z_1, Z_2)}{\sqrt{Var(Z_1)Var(Z_2)}}$

and $cov(Z_1, Z_2) = E(Z_1 Z_2) - E(Z_1)E(Z_2)$

$E(Z_1) = E(X_1 + X_2) = E(X_1) + E(X_2) = 0$

Similarly $E(Z_2) = 0$

Continued

Therefore, $cov(Z_1, Z_2) = E(Z_1 Z_2) = E((X_1+X_2)(X_2+X_3)) =$

$$E(X_1X_2) + E(X_1X_3) + E(X_2^2) + E(X_2X_3)$$

Since X_i's are pairwise uncorr. $\xrightarrow{i \neq j}$ $E(X_iX_j) = E(X_i)E(X_j) = 0$

$$\Rightarrow cov(Z_1, Z_2) = E(X_2^2) = var(X_2) + (\cancel{E(X_2)}^{0})^2 = \underline{1}$$

because X_i's are uncorr. $\Rightarrow Var(X_1+X_2) = Var(X_1) + Var(X_2) = 2$

Similarly $Var(X_2+X_3) = 2$

$$\Rightarrow \rho(Z_1, Z_2) = \frac{1}{\sqrt{2.2}} = \frac{1}{2}$$

b) define $Y_1 = X_1 + X_2$, $Y_2 = X_3 + X_4$

$$cov(Y_1, Y_2) = E(Y_1 Y_2) - \cancel{E(Y_1)}^{0}\cancel{E(Y_2)}^{0} = \cancel{E(X_1X_3)}^{0} + \cancel{E(X_1X_4)}^{0} + \cancel{E(X_2X_3)}^{0} + \cancel{E(X_2X_4)}^{0}$$

$$= 0$$

So $\rho(Y_1, Y_2) = 0$

Q4. we need to show that $cor(X+Y, X-Y) = 0$

$$cov(X+Y, X-Y) = E((X+Y)(X-Y)) - \underbrace{E(X+Y)}_{E(X)+E(Y)=2\mu}\underbrace{E(X-Y)}_{E(X)-E(Y)=0}$$

$$= E(X^2-Y^2) = E(X^2) - E(Y^2) =$$

$$(\mathrm{var}(X) + (E(X))^2) - (\mathrm{var}(Y) + (E(Y))^2) =$$

$$\sigma^2 + \mu^2 - (\sigma^2 + \mu^2) = 0 .$$

Q5) $\mathrm{cov}(X_i - \bar{X}, \bar{X}) = \mathrm{cov}(X_i, \bar{X}) - \mathrm{cov}(\bar{X}, \bar{X}) =$

$$\mathrm{cov}\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) - \mathrm{var}(\bar{X})$$

because X_i's are independent, $\mathrm{cov}(X_i, X_j) = 0$ when $i \neq j$

$$\Rightarrow \mathrm{cov}(X_i - \bar{X}, \bar{X}) = \underbrace{\frac{1}{n}\sum_{j=1}^{n} \mathrm{cov}(X_i, X_j)}_{(n-1)(0) + (1)\left(\frac{\sigma^2}{n}\right)} - \underbrace{\frac{\sigma^2}{n}}_{\frac{\sigma^2}{n}} = 0$$