Session 9.

Chapter 6.

Joint density function. X & $Y \longrightarrow$ R.V's

- $f_{X,Y}(x,y)$

$\star$ $P(a \leqslant X \leqslant b, c \leqslant Y \leqslant d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$

$\star$ we call $f_X(x)$ and $f_Y(y)$ Marginal densities.

To find marginal densities:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$$

X & Y are indep. R.V's if:

① $\left\{ \begin{array}{l} f_{X,Y}(x,y) = h_X(x)\,h_Y(y) \\ \quad\text{or } f_X(x)\,f_Y(y) \end{array} \right.$

and ② $\left\{ \begin{array}{l} \text{The range of } X \text{ \& } Y \\ \text{should be indep. of each other.} \end{array} \right.$

$\Rightarrow$ if X & Y are indep. we can say: $P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$ for any arbitrary events A & B.

To prove that X & Y are dependent we can find two arbitrary events like A and B such that:

$$P(X \in A, Y \in B) \neq P(X \in A)\,P(Y \in B)$$

1. $If\ X \sim U[0,1]\ and\ Y = 1 - X. Are\ X\ and\ Y\ independent?$

2. $If\ X1, X2\ and\ X3\ \ are\ uniformly\ distributed\ over\ [0,1], compute\ the\ probability\ that\ the\ largest\ one\ is$ *greater than sum of the other ones.*

3. $The\ joint\ pdf\ of\ X\ \&\ Y\ is\ given\ by\ f(x,y) = c(x^2+\frac{xy}{2})\ ;\ 0 \le X \le 1, 0 \le Y \le 2.$

a) $Find\ the\ constant\ c.$
b) $Find\ marginal\ pdf\ of\ X.$
c) $Find\ P\{X > Y\}.$
d) $P\{Y > \frac{1}{2} \mid X < \frac{1}{2}\}.$

- Conditional joint density function.

$$f_{X,Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx}$$

Simply we can define expected value of $g(x,y)$, where X & Y have the joint density function of $f_{X,Y}(x,y)$ in this way: $E(g(x,y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\,dy.$

* we read this property in chapter 7. but it is worthy to solve some problems regarding $E(g(x,y))$.

e.g: in Q3, find $E(XY)$

- simply $E(XY) = \iint_{\substack{0<x<1 \\ 0<y<2}} xy \cdot \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy =$

$$\int_0^1 \frac{6}{7}\left(\frac{x^3y^2}{2} + \frac{x^2}{2}\,\frac{y^3}{3}\right)\Big|_0^2 dx = \frac{6}{7}\left(\frac{2x^4}{4} + \frac{4x^3}{9}\right)\Big|_0^1 = \frac{17}{21}$$

4. The bivariate random variable (X, Y) has the joint pdf

$$f_{XY}(x, y) = 2\exp[-(x+y)], \quad 0 < x < y, \; y > 0.$$

Evaluate the following expressions.

(a) $P(Y < 1 | X < 1)$.

(b) $P(Y < 1 | X = 0.5)$.

(c) $P(0.25 < Y < 0.75 | X = 1)$.

5. *The joint pdf of X and Y is given by* $f(x, y) = e^{-y}/y; 0 < x < y, 0 < y < \infty$.

Find $f(x|y)$

6. Three points $X1$,$X2$,$X3$ are selected at random on a line L. What is the probability that $X2$ lies between $X1$ and $X3$?