Session 3
(Tutorial Notes)

Topic: Conditional Probability

1. Why we need to know how to compute "conditional Probability"?

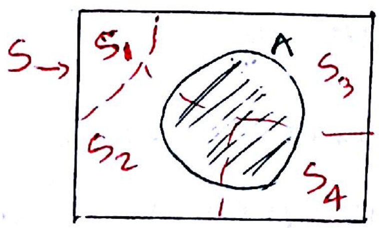
First: In nature events are dependant and because of this, occurrence of an event can change probability of another event. For modelling this new probability, we need to compute conditional probability:

A & B are dependant $\Rightarrow \begin{cases} P(A|B) \neq P(A) \\ P(B|A) \neq P(B) \end{cases}$ $\quad \begin{cases} P(A|B) = \dfrac{P(A \cap B)}{P(B)} \\ P(B|A) = \dfrac{P(A \cap B)}{P(A)} \end{cases}$

Trap 1: If you have two events: A & B, and you know: 1. A&B are dependant

2. Event B has happened. In this case you can not use $P(A)$ as probability of event A anymore!

Second: Sometimes we only have one event and we want to compute probability of that event, but ~~one~~ for some reasons, which we will see in this session, we can not find probability in regular way (Counting). In this case we need to divide the ~~event~~ sample space into some "Mutually Exclusive" sub-events:



$P(A) = P((S_1 \cap A) \cup (S_2 \cap A) \cup \dots (S_n \cap A))$

Mutually Exclusive $= P(S_1 \cap A) + \dots + P(S_n \cap A)$

$= P(A|S_1)P(S_1) + \dots + P(A|S_n)P(S_n)$

$= \sum_{i=1}^{n} P(A|S_i)P(S_i)$

Trap 2: If chosen sub-events are not "Mutually Exclusive", we can not write union of them as summation of probability of each one.

Example: A = Two heads when coin is flipped two times

S_1 = First flip is head
S_2 = " " " Tail

$S_3 = HH$ (first, second)

$S_4 = TH \quad S_5 = HT \quad S_6 = TT$

$S_1 \cup S_2 \cup \dots \cup S_6 = S$ ✓ First condition

But: $S_1 \cap S_3 \neq \phi$ ✗

$$P(A) = \frac{1}{4} \neq \underbrace{P(A|S_1)}_{\frac{1}{2}} \times \underbrace{P(S_1)}_{\frac{1}{2}} + \underbrace{P(A|S_2)}_{0} \times \underbrace{P(S_2)}_{\frac{1}{2}} + \underbrace{P(A|S_3)}_{1} \times \underbrace{P(S_3)}_{\frac{1}{4}} + \underbrace{P(A|S_4)}_{0} \times \underbrace{P(S_4)}_{\frac{1}{2}}$$

$$+ \underbrace{P(A|S_5)}_{0} \times \underbrace{P(S_5)}_{\frac{1}{2}} + \underbrace{P(A|S_6)}_{0} \times \underbrace{P(S_6)}_{\frac{1}{2}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

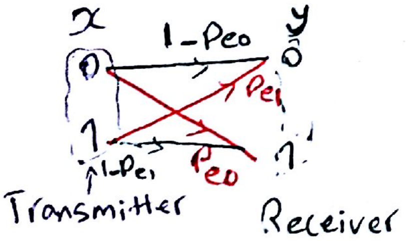
Trap 3: Union of all sub-events should be whole sample space (S).

2. Different types of problems:

a) We have $P(B|A)$, but we want: $P(A|B)$ or $P(A)$ or $P(B)$

Problem 1: We have a communication system in which we send one bit $\{0,1\}$, through a channel, with $\{P_0, P_1\}$. In channel with probability P_{e0}, $0 \to 1$ and P_{e1}, $1 \to 0$

find:



P_0: Probability of sending 0 $\quad P(x=0)$

P_1: " " " 1 $\quad P(x=1)$

P_{e0}: 0 is sent $\Longrightarrow$ Probability of receiving 1

$$P(y=1 | x=0) = 1 - P(y=0 | x=0)$$

P_{e1}: $P(y=0 | x=1) = 1 - P(y=1 | x=1)$

Problem 2: (Winter 2012)
Midterm

A student has asked his supervisor for a letter of recommendation for an award. He estimates that there is a 90% chance that he will get the award if he receives a strong recommendation, a 50% chance if he receives a moderately good recommendation, and a 10% chance if he receives a weak recommendation. He further estimates that the probabilities that the recommendation will be strong, moderate, and weak are 0.7, 0.2, and 0.1, respectively.

(a) (5 points) How certain is he that he will receive the award?

(b) (10 points) Given that he does receive the award, how likely should he feel that he received a strong recommendation?

(c) (10 points) Given that he does not receive the award, how likely should he feel that he received a weak recommendation?

b) We want to find probability of an event, but desired outcomes can not be counted and experiment restarts after some trials.

Problem 3:

A and B roll a pair of dice in turn, with A rolling first. A's objective is to obtain a sum of 6, and B's is to obtain a sum of 7. The game ends when either player reaches his or her objective, and that player is declared the winner.

(a) Find the probability that A is the winner.

Problem 4:

Two players take turns shooting at a target, with each shot by player i hitting tl target with probability p_i, $i = 1, 2$.

Suppose that the shooting ends when the target has been hit twic Let m_i denote the mean number of shots needed for the first hit when player i shoo first, $i = 1, 2$. Also, let P_i, $i = 1, 2$, denote the probability that the first hit is b player 1, when player i shoots first.

(a) Find m_1 and m_2. Ch 7

(b) Find P_1 and P_2.

For the remainder of the problem, assume that player 1 shoots first.

(c) Find the probability that the final hit was by 1.

(d) Find the probability that both hits were by 1.

(e) Find the probability that both hits were by 2.

(f) Find the mean number of shots taken. Ch 7

Problem 5: A, B, and C are evenly matched tennis players. Initially A and B play a set, and the winner then plays C. This continues, with the winner always playing the waiting player, until one of the players has won two sets in a row. That player is then declared the overall winner. Find the probability that A is the overall winner.

c) Conditioning helps us to find a recursive formula for finding probability

Problem 6:

Independent trials that result in a success with probability p and a failure with probability $1 - p$ are called *Bernoulli trials*. Let P_n denote the probability that n Bernoulli trials result in an even number of successes (0 being considered an even number). Show that

$$P_n = p(1 - P_{n-1}) + (1 - p)P_{n-1} \quad n \geq 1$$

and use this formula to prove (by induction) that

$$P_n = \frac{1 + (1 - 2p)^n}{2}$$

Problem 7 **(The Ballot Problem)** In an election, candidate A receives n votes, and candidate B receives m votes where $n > m$. Assuming that all orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n - m)/(n + m)$.

Problem 8:

Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0?

Problem 9:

A particle moves among $n + 1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $q = 1 - p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the probability that all states have been visited by time T.