Session 11

Conditional expectation

discrete R.V

$$E(X \mid Y=y) = \sum_{x} x \, P_{X|Y}(x|y)$$

$$E(g(X) \mid Y=y) = \sum_{x} g(x) \, P_{X|Y}(x|y)$$

continuous R.V

$$E(X \mid Y=y) = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\, dx$$

$$E(g(X) \mid Y=y) = \int_{-\infty}^{+\infty} g(x) \, f_{X|Y}(x|y)\, dx$$

and also we have: $Var(X \mid Y) = E(X^2 \mid Y) - (E(X \mid Y))^2$

Q1. *The joint PDF of X and Y is given by* $f(x,y) = (2m+6)x^m y;\ 0 < y < x < 1$
Find $Var(Y^2 \mid X = x)$.

Moment Generating Function (MGF)

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} \cdot p(x) & ; \quad \text{if } X \text{ is discrete} \\ \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx & ; \quad \text{if } X \text{ is continuous.} \end{cases}$$

Note 1. $E(X^n) = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$, for example, $\begin{cases} E(X) = M_X'(0) \\ E(X^2) = M_X''(0) \end{cases}$

Note 2. MGF of the sum of indep. r.v's equals to the product of individual MGFs, i.e., if $X_1, X_2, \dots, X_n$ are indep, then:

$$M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t)$$

Q2. *If Z is a standard normal r.v. and if Y is defined by $Y = a + bZ + cZ^2$, then find $\rho(x, y)$.*

Q3. *Let* $M_X(t) = e^{2e^{t}-2}$ *and* $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$. *If X and Y are independent, find $E(XY)$.*

Limit Theorems

* The central Limit theorem (CLT)

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed R.V's, each having mean "μ" and variance σ^2.

Then, $Y = \dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \overset{n>30}{\sim} N(0,1)$

In other words, as n gets larger the r.v Y, which is defined based on X_i's, would have Standard normal distribution.

Q4. *Let* $X1, X2, \dots, X20$ *be independent Poisson R.V.s with mean* 1 *and variance* 1.
Use CLT theorem to approximate $P\{\sum_{i=1}^{20} Xi > 15\}$