Problem I:

We want to prove by induction:

when we want to prove one expression by induction we have some steps:

1. we should test that for some n & m

Suppose: $n=2$ $m=1$

AAB only possible way in which A is always ahead $\Rightarrow P_{2,1} = \frac{1}{3}$ It should be equal to equation which we want to prove, where: $n=2$, $m=1$

$P_{n,m} = \frac{n-m}{n+m} \xRightarrow[m=1]{\text{Putting } n=2} P_{2,1} = \frac{1}{3}$ → Equal

So first step is correct ✓

2. we assume that formula is correct for $(n-1,m)$, $(n,m-1)$, $(n-1,m-1)$

So we assume that:
$$\begin{cases} P_{n-1,m} = \dfrac{(n-1)-m}{(n-1)+m} \\ P_{n,m-1} = \dfrac{n-(m-1)}{n+(m-1)} \end{cases}$$

Note that it is only an assumption.

before going into last step it is good to know meaning of $P_{n-1,m}$ & $P_{n,m-1}$:

$P_{n-1,m}$ = Probability that A is always ahead when A has $(n-1)$ votes and B has (m) votes

$P_{n,m-1}$ = 〃 〃 〃 〃 〃 〃 〃 〃 〃 n 〃 〃 〃 〃 $(m-1)$ 〃

3. we should somehow use our assumption to show that $P_{n,m} \stackrel{?}{=} \frac{n-m}{n+m}$

by conditioning on last vote:

$$P_{n,m} = \bigcirc\, P(\text{Last vote for A}) + \triangle\, P(\text{Last vote for B})$$

$P(\text{Last vote for A}) = \frac{n}{n+m}$

$P(\text{Last vote for B}) = \frac{m}{n+m}$ → # of B's votes

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$\oslash$: $\underbrace{\_\ \_\ \_ \cdots \_\ \_}_{\substack{n+m-1 \\ \text{votes}}}$ $\underset{\substack{\uparrow \\ \text{Last} \\ \text{vote}}}{\underline{A}}$

$n+m-1$ votes $\Rightarrow$ $(n-1)$ for A, m for B

If we want to A be always ahead, he also should be ahead in $n+m-1$ first votes which $(n-1)$ of them are for A & m are for B

$n-1$ because last vote is for A

$$\oslash = P(\text{A always ahead} \mid \text{Last one for A}) = P(\text{A always ahead when A has } \underline{n-1} \text{ votes \& B has } \underline{m} \text{ votes})$$

$$= P_{(n-1),m} \leftarrow \text{refer to second Step}$$

$\triangle$: $\underbrace{\_\ \_\ \_ \cdots \_\ \_}_{\substack{n+m-1 \\ \text{votes}}}$ $\underline{B}$

n for A

$(m-1)$ " B

$m-1$: Last vote is for B

$$\triangle = P_{n,(m-1)}$$

$$P_{n,m} = P_{(n-1),m} \frac{n}{n+m} + P_{n,(m-1)} \frac{m}{n+m}$$

$\vdots$

$$\boxed{P_{n,m} = \frac{n-m}{n+m}} \checkmark$$

☆ Note: If we condition on first vote we could not write as follows:

$$P_{n,m} \neq P_{(n-1,m)} \underset{\substack{\text{First vote} \\ \text{for A}}}{\frac{n}{n+m}} + 0 \times \underset{\substack{\text{First vote} \\ \text{for B}}}{\frac{m}{n+m}}$$

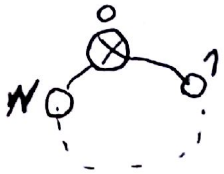
why It is wrong?

In this condition A can not be always ahead ~~because A~~

problem 2:

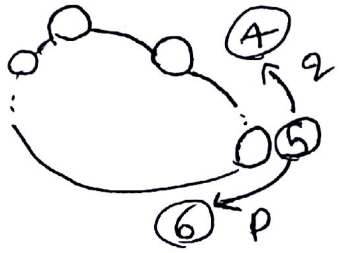
★ when question wants probability of reaching something before something else, gambler's ruin formula can help but first we should make question similar to gambler's ruin:

In this question starting point is 0 :

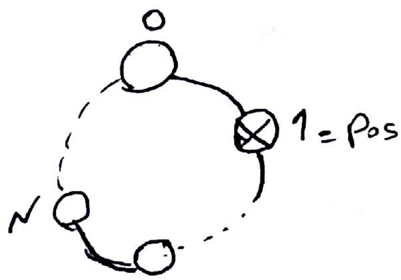


we want to find probability of <u>visiting</u> all spots <u>before</u> going back to 0

At each step, particle moves clockwise with probability p and counterclockwise with probability q :



If we know that particle chose clockwise step for first movement:



we want to find: $P($ Pos $= N$ before Pos $= 0$ | we start at Pos $= 1)$

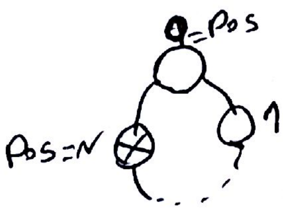
Pos → money ⇒ $P(N$ units before 0 unit | start with 1 unit$)$

$$= \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N}$$

q: probability that Pos decrease

p: 〃 〃 〃 increase

If we know that:



$P($Pos $= 1$ before Pos $= 0$ | Pos $= N$ at first$)$

★ Pos $= 1$ ⇒ N units

Pos $= 0$ ⇒ 0 〃

Pos $= N$ ⇒ 1 〃

★ Pos → Pos+1 : q ★★

Pos → Pos−1 : p

$$\frac{1 - \frac{p}{q}}{1 - \left(\frac{p}{q}\right)^N}$$

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Put all together:

$$P(A) = \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} p + \frac{1-\frac{p}{q}}{1-\left(\frac{p}{q}\right)^N} q$$

First move clockwise

First move counter clockwise

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