ECE316- Probability and Random Variables Winter 2007
In-term Exam 2

Time: 1 Hour 15 minutes

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INSTRUCTIONS

- Enter your name, student ID number, e-mail address and sign in the space provided at the bottom of this page.
- The exam is **1 hour and 15 mins** long.
- DO ALL QUESTIONS. The weightage of each question is indicated.
- Some useful results and formulae are given on the last page.
- You are **not** allowed the use of calculators or crib sheets.
- Unless specifically mentioned to the contrary show all relevant work in the space provided. You may use the extra blank pages and the back of each page if necessary.

Name:
Student ID #:
E-mail :
Signature:

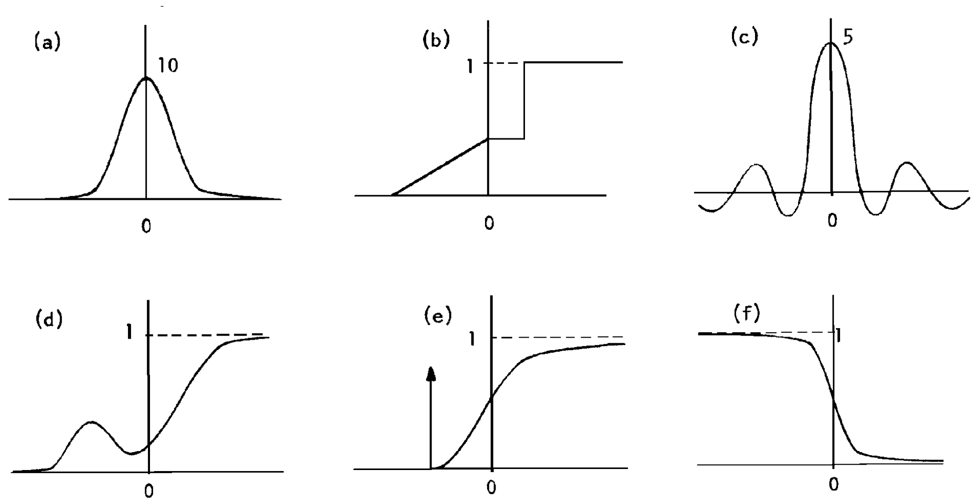
PART I

(26 %)

These are short problems. You must give a justification for your answers.

1. (12pts) For each of the figures shown below indicate whether it could be i) A probability distribution function , ii) A probability density function (pdf), iii) Either a distribution function or a pdf, or iv) Neither.

   Note the values (except for the origin) on the X-axis are not specified.



Answers:

a)

b)

c)

d)

e)

f)

2. (4 points) Let A, B and C be three arbitrary events on $(\Omega, \mathcal{F}, P)$. Show that

$$P(A|B \cap C)P(B|C) = P(A \cap B|C)$$

3. (4 points) Let $F(x)$ be a probably distribution function of a r.v. defined on $[a, b]$. State its properties.

4. (6 points) Let $X(\omega)$ be a normal r.v. with $E[X] = 1$ and $E[X^2] = 2$.

   Using the table given at the back of this paper find:

   a) $P(X < 3)$

   b) $P(0 < X < 3)$

   c) $P(X \leq -2)$

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PART II

For the problems in this section show all calculations in the space provided or at the back of the page. Justify all your answers.

Problem 1: (24%)
a) Let $X(\omega)$ be a discrete r.v. taking values $\{0, 1, 2, \ldots, \}$. Supose $X(\omega)$ has a Poisson distribution with parameter $\lambda > 0$.

Find the probability that $X(\omega)$ is even.

b) You have N coins in your pocket where N is Poisson with parameter λ. You toss each coin once and the probability of coming heads is p. Show that the total number of heads is also Poisson with parameter λp.

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Problem 2: (24%)

Suppose that the amplitude of a signal is random and given by a random variable $X(\omega)$ with pdf given by

$$p_X(x) = e^{-2|x|}, \quad -\infty < x < \infty$$

Suppose the signal is passed through a squae-law detector whose output $Y = 0.5X^2$.

a) Find the probability density function (pdf) of $Y(\omega)$.

b) Find the mean of $Y(\omega)$.

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Problem 3: (26%)

Let $X(\omega)$ be a continuous r.v. with pdf given by:

$$P_X(x) = C(x - x^2)$$

where $\alpha \le x \le \beta$ and $C > 0$.

a) What are the possible values of α and β? Find the value of C.

b) Find the mean and variance of the r.v.

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Some useful formulae and notes

Abbreviations used: Pr. = Probability, r.v. = Random Variable, c.d.f. = Cumulative Distribution Function (denoted F(x)), p.d.f = Probability density function (for continuous distributions) (usually denoted p(x)), p.m.f= Probability mass function (for discrete distributions) (usually denoted $p_k = \Pr(X = x_k)$.

For right continuous functions $G(x)$ the derivative is defined from the right i.e.:

$$g(x) = \frac{dG(x)}{dx} = \lim_{\Delta \to 0} \frac{G(x+\Delta) - g(x)}{\Delta}$$

when it exists.

Some elementary formulae

: If X is a discrete-r.v. then the $k-th$ moment of X:

$$\mathbf{E}[X^k] = \sum_n x_n^k p_n$$

where $p_n = \Pr(X = x_n)$.

If X is continuous then:

$$\mathbf{E}[X^k] = \int_{\Re} x^k dF(x) = \int_{\Re} x^k p(x) dx$$

where $p(x) = \frac{dF(x)}{dx}$is the p.d.f. and $F(x)$ is the c.d.f.

A discrete r.v. X is to be Binomial (n,p) often denoted $Bin(n,p)$ if:

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}. \quad k = 0, 1, \cdots, n$$

A discrete r.v. is said to be Poisson with parameter λ if:

$$\Pr(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \cdots$$

The distribution function of a standard Normal (or Gaussian) r.v. is given by:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \quad -\infty < x < \infty$$

Some other elementary formulae

Given a collection of sets $\{A_i\}$ then De-Morgan's law states:

$$\left(\cup_{k=1}^{n} A_k\right)^c = \cap_{k=1}^{n} A_k^c$$

where $A_k^c = \Omega - A$ (the complement of A_k.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \quad if \ |r| < 1$$

Integration by parts:

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

Jacobian formula

Let $\mathbf{X}$ be a $\Re$ valued r.v. and let $\mathbf{Y} = f(\mathbf{X})$ be a $\Re$ valued r.v. and $f(.)$ be monotone mapping.

Then:

$$P_Y(y) = \frac{1}{|f'(f^{-1}(y))|} P_X(f^{-1}(y))$$

$\phi(x)$ vs. x , The Normalized Gaussian Distribution Function

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998
3.50	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.60	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.70	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000