ECE 316-Solutions of Mid Term 2

April 5, 2009

PART I

1.

a)

It can be a Probability Density Function. It cannot be a probability distribution function because it doesn't tends to 1 as $x- > \infty$ and it decreases.

b)

It can be a Probability distribution function. It cannot be a probability density function because the area inside is not equal to 1. In fact the area is infinity.

c)

Neither. Probability Density Function or a Probability Distribution Function cannot take negative values.

d)

Neither. The area is infinity, so not a density function. Not a Probability distribution function because the function decreases in between.

e)

Neither. The area is infinity, so not a density function. Not a Probability distribution function because after the spike, the function starts from zero, in other words it decreases.

f)

Neither. The area is infinity, so not a density function. Not a Probability distribution function because the function decreases.

2.

$$\begin{aligned}
P(A|B \cap C)P(B|C) &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \frac{P(B \cap C)}{P(C)} \\
&= \frac{P(A \cap B \cap C)}{P(C)} \\
&= P(A \cap B|C)
\end{aligned}$$

3.

Properties of F(x)

i)

$$\begin{aligned}
F(x) = 1, &\quad x \geq b \\
F(x) = 0, &\quad x \leq a
\end{aligned}$$

ii) $\alpha > \beta \implies F(\alpha) \geq F(\beta)$. In other words F(x) is not decreasing.
iii) $F(x)$ is right continuous.

4.

Mean $m = E(X) = 1$, $E(X^2) = 2$. Therefore $Var(X) = E(X^2) - E(X)^2 = 2 - 1^2 = 1$, and $\sigma = \sqrt{1} = 1$. Let Z is a standard normal variable(i.e. mean= 0, and standard deviation=1).
a)

$$\begin{aligned}
P(X < 3) &= P(Z < \frac{3-m}{\sigma}) \\
&= P(Z < \frac{3-1}{1}) \\
&= P(Z < 2) \\
&= 0.9773
\end{aligned}$$

b)

$$\begin{aligned}
P(0 < X < 3) &= P\left(\frac{0-1}{1} < Z < \frac{3-1}{1}\right) \\
&= P(-1 < Z < 2) \\
&= P(Z < 2) - P(Z \leq -1) \\
&= P(Z < 2) - P(Z \geq 1) \\
&= p(Z < 2) - (1 - P(Z < 1)) \\
&= 0.9773 - (1 - 0.8413) \\
&= 0.8186
\end{aligned}$$

c)

$$
\begin{aligned}
P(X \leq -2) &= P(Z \leq \frac{-2-1}{1}) \\
&= P(Z \leq -3) \\
&= P(Z \geq 3) \\
&= 1 - P(Z < 3) \\
&= 1 - 0.9987 \\
&= 0.0013
\end{aligned}
$$

PART II

Problem 1

a) X has a Poisson distribution with parameter $\lambda > 0$. Therefore

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$
\begin{aligned}
P(X \text{ is even}) &= P(X = 0) + P(X = 2) + P(X = 4) + \cdots \\
&= e^{-\lambda}\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} \cdots\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
e^{\lambda} &= 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} \cdots \\
e^{-\lambda} &= 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} \cdots
\end{aligned}
$$

Therefore

$$\frac{e^{\lambda} + e^{-\lambda}}{2} = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} \cdots$$

and

$$P(X \text{ is even}) = e^{-\lambda}\left(\frac{e^{\lambda} + e^{-\lambda}}{2}\right) = \frac{1 + e^{-2\lambda}}{2}$$

b) Let N denotes the total number of coins and N_H the number of heads. Given that N is Poisson with parameter λ, i.e.

$$P(N = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$
\begin{aligned}
P(N_H = m) &= \sum_{k=0}^{\infty} P(N_H = m | N = k) P(N = k) \\
&= \sum_{k=m}^{\infty} P(N_H = m | N = k) P(N = k) \qquad \text{Since } P(N_H = m|N = k) = 0 \text{ for } k < m \\
&= \sum_{k=m}^{\infty} \binom{k}{m} p^m (1-p)^{k-m} \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \frac{e^{-\lambda p}(\lambda p)^m}{m!} \sum_{k=m}^{\infty} \frac{e^{-\lambda(1-p)}(\lambda(1-p))^{k-m}}{(k-m)!} \\
&= \frac{e^{-\lambda p}(\lambda p)^m}{m!} \sum_{z=0}^{\infty} \frac{e^{-\lambda(1-p)}(\lambda(1-p))^{z}}{z!} \\
&= \frac{e^{-\lambda p}(\lambda p)^m}{m!}
\end{aligned}
$$

Problem 2

The pdf of X is given by

$$p_X(x) = e^{-2|x|}, \qquad -\infty < x < \infty$$

The signal is passed through a square law detector whose output is $Y = 0.5X^2$

a)

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(0.5X^2 \leq y) \\
&= P(X^2 \leq 2y) \\
&= P(-\sqrt{2y} \leq X \leq \sqrt{2y}) \\
&= \int_{-\sqrt{2y}}^{\sqrt{2y}} e^{-2|x|} dx \\
&= \int_{-\sqrt{2y}}^{0} e^{-2|x|} dx + \int_{0}^{\sqrt{2y}} e^{-2|x|} dx \\
&= \int_{-\sqrt{2y}}^{0} e^{2x} dx + \int_{0}^{\sqrt{2y}} e^{-2x} dx \\
&= \left[\frac{e^{2x}}{2}\right]_{-\sqrt{2y}}^{0} + \left[\frac{e^{-2x}}{-2}\right]_{0}^{-\sqrt{2y}} \\
&= 1 - e^{-2\sqrt{2y}}
\end{aligned}
$$

Therefore

$$f_Y(y) = \frac{d}{dy}(1 - e^{-2\sqrt{2y}}) = \sqrt{\frac{2}{y}} e^{-2\sqrt{2y}}, \quad y \geq 0$$

Alternatively you can find $f_Y(y)$ by using the Leibniz Integral Rule which is

$$\frac{d}{dz}\int_{a(z)}^{b(x)} f(x,z)dx = \int_{a(z)}^{b(x)} \frac{d}{dz} f(x,z)dx + f(b(z),z)\frac{db}{dz} - f(a(z),z)\frac{da}{dz}$$

Therefore

$$f_Y(y) = \frac{d}{dy}\int_{-\sqrt{2y}}^{\sqrt{2y}} e^{-2|x|}dx = e^{-2\sqrt{2y}}\frac{1}{\sqrt{2y}} - e^{-2\sqrt{2y}}\frac{-1}{\sqrt{2y}} = \sqrt{\frac{2}{y}}e^{-2\sqrt{2y}}, \quad y \geq 0$$

b)

$$E(Y) = \int_{y=0}^{\infty} y\sqrt{\frac{2}{y}}e^{-2\sqrt{2y}}dy$$

Substituting $x = 2\sqrt{2y}$ and therefore $dx = \sqrt{\frac{2}{y}}dy$ we have

$$\begin{aligned} E(Y) &= \frac{1}{8}\int_{0}^{\infty} x^2 e^{-x}dx \\ &= \frac{1}{4} \end{aligned}$$

Note, if you are not asked to find put the density of Y, you can find out $E(Y)$ as

$$\begin{aligned} E(Y) &= E(0.5X^2) \\ &= \int_{-\infty}^{\infty} 0.5x^2 e^{-2|x|}dx \\ &= \frac{1}{4} \end{aligned}$$

Problem 3

The pdf of X is given by

$$P_X(x) = C(x - x^2), \quad \alpha \leq x \leq \beta, \quad C > 0$$

a) Since pdf is non negative

$$\begin{aligned} x - x^2 &\geq 0 \\ x(x-1) &\leq 0 \\ &\implies 0 \leq x \leq 1 \\ &\implies 0 \leq \alpha < \beta \leq 1 \end{aligned}$$

To find out the value of C

$$\begin{aligned} \int_{\alpha}^{\beta} C(x - x^2)dx &= 1 \\ C\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{\alpha}^{beta} &= 1 \\ C\left(\frac{\beta^2 - \alpha^2}{2} - \frac{\beta^3 - \alpha^3}{3}\right) &= 1 \\ C &= \frac{1}{\frac{\beta^2-\alpha^2}{2} - \frac{\beta^3-\alpha^3}{3}} \end{aligned}$$

Similarly

$$\begin{aligned}
E(X) &= \int_{\alpha}^{\beta} xC(x - x^2)dx \\
&= C\left(\frac{\beta^3 - \alpha^3}{3} - \frac{\beta^4 - \alpha^4}{4}\right) \\
&= \left(\frac{1}{\frac{\beta^2-\alpha^2}{2} - \frac{\beta^3-\alpha^3}{3}}\right)\left(\frac{\beta^3 - \alpha^3}{3} - \frac{\beta^4 - \alpha^4}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
E(X^2) &= \int_{\alpha}^{\beta} x^2C(x - x^2)dx \\
&= C\left(\frac{\beta^4 - \alpha^4}{4} - \frac{\beta^5 - \alpha^5}{5}\right) \\
&= \left(\frac{1}{\frac{\beta^2-\alpha^2}{2} - \frac{\beta^3-\alpha^3}{3}}\right)\left(\frac{\beta^4 - \alpha^4}{4} - \frac{\beta^5 - \alpha^5}{5}\right)
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= \left(\frac{1}{\frac{\beta^2-\alpha^2}{2} - \frac{\beta^3-\alpha^3}{3}}\right)\left(\frac{\beta^4 - \alpha^4}{4} - \frac{\beta^5 - \alpha^5}{5}\right) - \left(\left(\frac{1}{\frac{\beta^2-\alpha^2}{2} - \frac{\beta^3-\alpha^3}{3}}\right)\left(\frac{\beta^3 - \alpha^3}{3} - \frac{\beta^4 - \alpha^4}{4}\right)\right)^2
\end{aligned}$$