ECE316- Probability and Random Variables Winter 2009
In-term Exam 1

Time: 1 Hour 15 minutes

R. Mazumdar **Date : February 2, 2009**

INSTRUCTIONS

- Enter your name, student ID number, e-mail address and sign in the space provided at the bottom of this page.
- The exam is **1 hour and 15 mins** long.
- DO ALL QUESTIONS. The weightage of each question is indicated.
- Some useful results and formulae are given on the last page.
- You are **not** allowed the use of calculators or crib sheets.
- Unless specifically mentioned to the contrary show all relevant work in the space provided. You may use the extra blank pages and the back of each page if necessary.

Name:
Student ID #:
E-mail :
Signature:

PART I

(20 %)

These are short problems. For problems involving TRUE or FALSE answers it is not necessary to justify your answers. Only correct answers count- no partial credit.

1. (4pts) Let A and B be two events defined on $(\Omega, \mathcal{F}, P)$: Then:

   a) $\Pr(A) + \Pr(B^c) \leq 1$.

   Answer: True False

   b) $\Pr(A \cup B^c) \geq \max\{\Pr(A), 1 - \Pr(B)\}$

   Answer: True False

2. (6pts) Suppose A, B, C are three events defined on the same probability space.

   a) Find $P(A \cup B \cup C)$ if $A, B,$ and C are mutually independent

b) If A and B are independent, show that A^c and B^c are independent.

3. (4 pts) Let A and B be two events with $\Pr(A) > 0$ and $\Pr(B) > 0$.
   Show that if $\Pr(A|B) \geq \Pr(A)$ then $\Pr(B|A) \geq \Pr(B)$.

4. (6pts) Express the following probabilities in terms of $P(A), P(B)$ and $P(A \cap B)$

   a) The event A occurs and B does not occur.

   b) Exactly one of A or B occurs.

   c) Neither A nor B occur.

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PART II

For the problems in this section show all calculations in the space provided or at the back of the page. Justify all your answers.

Problem 1: (18 %)

Two numbers X and Y are selected at random between 0 and 1. Let the events A, B, and C be defined as follows: $A = \{X > \frac{1}{2}\}, B = \{Y < \frac{1}{2}\}$ and $C = \{X < \frac{1}{2}\ and\ Y < \frac{1}{2}\} \cup \{X > \frac{1}{2} and Y > \frac{1}{2}\}$

a) Are A and B independent? Why?

b) Are A and C independent? Why?

c) Are A, B and C independent? Why?

d) Find the probability that only one of the events A or B occur.

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Problem 2: (**20%**)

a) Let A and B be two arbitrary events. Show that:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

b) An urn contains M balls out of which only m are good. Two balls are drawn from the urn without replacement.

i) Find the probability that the second ball is "good" given that the first ball is "bad".

ii) Find the probability that the second ball is good.

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Problem 3: (**20** %)

Let $\{A_i\}_{i=1}^n$ denote a collection of events on a probability space $(\Omega, \mathcal{F}, P)$
a) Show that

$$P(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n P(A_i^c)$$

b) Show the following formula holds:

$$P(\cap_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) \quad + \quad \sum_{i<j<k} P(A_j \cup A_j \cup A_k) - \cdots$$
$$+ \cdots (-1)^{n+1} P(A_1 \cup A_2 \cdots \cup A_n)$$

c) Show for any events A, B and C with $P(C) > 0$ show that:

$$P(A|C) = P(A|BC)P(B|C) + P(A|B^cC)P(B^c|C)$$

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Problem 4: (22%)

a) Consider 3 events A, B and C. It is known that A is independent of B and A is independent of C.

Show that A is independent of $B \cup C$ if and only if A is independent of $B \cap C$.

b) Consider an urn with two types of coins. One is a fair coin which has a probability $\frac{1}{2}$ of giving a Heads when tossed. The other is a biased coin that gives Heads with a probability of $\frac{1}{3}$ when tossed. There are a total of n fair coins and n biased coins.

a) A coin is chosen at random and tossed and gives a Heads. What is the probability that the coin is fair?

a) Two coins are chosen and tossed and both fall heads. What is the probability that both are biased?

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Some useful formulae and notes

Some other elementary formulae

Given a collection of sets $\{A_i\}$ then De-Morgan's law states:

$$\left(\cup_{k=1}^{n} A_k\right)^c = \cap_{k=1}^{n} A_k^c$$

where $A_k^c = \Omega - A$ (the complement of A_k.

Probability axioms

1. $P(\Omega) = 1$, $P(\phi) = 0$, and for any event A $0 \leq P(A) \leq 1$.
2. The collection of events is a $\sigma - field$ i.e. if $A \in \mathcal{F}$ then $A^C \in \mathcal{F}$ and for any countable collection of events $A_i \in \mathcal{F}$, $\cup_i A_i \in \mathcal{F}$.
3. If $\{A_i\}$ is a countable collection of disjoint events then: $P(\cup_n A_n)) = \sum_n P(A_n)$. If the $A_i's$ are not disjoint then $P(\cup_i A_i) \leq \sum_i P(A_i)$

Exclusion-Inclusion Formula Given events $\{A_i\}_{i=1}^n$

$$P(\cup_{i=1}^n A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) \quad + \quad (-1)^{k+1} \sum_{i_1 < i_2 \ldots < i_k} P(A_{i_1} \cdots A_{i_k}) + \cdots + (-1)^{n+1} P(A_1 A_2 \cdots A_n)$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad if \ |r| < 1$$