ECE 316-Solutions of Mid Term 1

March 16, 2009

PART I

1.

a) False
A and B^c are two events and each of them can take value between 0 and 1. So their sum can be more than 1.
b) True
$Pr(A \cup B^c) \geq \max(Pr(A), 1 - Pr(B))$ is equivalent to $Pr(A \cup B^c) \geq \max(Pr(A), Pr(B^c))$ which is true.

2.

a) A, B and C are mutually independent implies that

$$\begin{aligned}
P(A \cap B) &= P(A)P(B) \\
P(B \cap C) &= P(B)P(C) \\
P(C \cap A) &= P(C)P(A) \\
P(A \cap B \cap C) &= P(A)P(B)P(C)
\end{aligned}$$

Therefore

$$\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \\
&= P(A) + P(B) + P(C) - P(A)P(B) - P(B)P(C) - P(C)P(A) + P(A)P(B)P(C)
\end{aligned}$$

b)
A and B are independent implies that

$$P(A \cap B) = P(A)P(B)$$

$$
\begin{aligned}
P(A^c \cap B^c) &= P(A \cup B)^c \\
&= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= (1 - P(A)) - P(B)(1 - P(A)) \\
&= (1 - P(A))(1 - P(B)) \\
&= P(A^c)P(B^c)
\end{aligned}
$$

Therefore A^c and B^c are independent.

3.

A and B be two events with $P(A) > 0$ and $P(B) > 0$

$$
\begin{aligned}
&P(A|B) \geq P(A) \\
\implies &\frac{P(A \cap B)}{P(B)} \geq P(A) \\
\implies &\frac{P(A \cap B)}{P(A)} \geq P(B) \\
\implies &P(B|A) \geq P(B)
\end{aligned}
$$

4.

a)

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

b)

$$P(AB^c \cup A^cB) = P(AB^c) + P(AB^c) - P(AB^cA^cB) = P(A) + P(B) - 2P(AB)$$

c)

$$P(A \cup B)^c = 1 - P(A) - P(B) + P(A \cap B)$$

PART II

Problem 1:

a) X and Y are selected randomly. A depends on X only and B depends on Y only. So A and B are independent.

b)

$$
\begin{aligned}
P(A) &= P(X > \frac{1}{2}) = \frac{1}{2} \\
P(C) &= P\left(\left((X < \frac{1}{2}) \text{ and } (Y < \frac{1}{2})\right) \cup \left((X > \frac{1}{2}) \text{ and } (Y > \frac{1}{2})\right)\right) \\
&= P\left((X < \frac{1}{2}) \text{ and } (Y < \frac{1}{2})\right) + P\left((X > \frac{1}{2}) \text{ and } (Y > \frac{1}{2})\right) \\
&\quad - P\left(\left((X < \frac{1}{2}) \text{ and } (Y < \frac{1}{2})\right) \cap \left((X > \frac{1}{2}) \text{ and } (Y > \frac{1}{2})\right)\right) \\
&= \frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2} - 0 \\
&= \frac{1}{2} \\
P(A \cap C) &= P\left((X > \frac{1}{2}) \text{ and } (Y > \frac{1}{2})\right) \\
&= \frac{1}{2}.\frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}
$$

Therefore

$P(A \cap C) = P(A)P(C)$

A and C are independent

c) $A = (X < \frac{1}{2})$, $B = (Y > \frac{1}{2})$ and $C = \big(((X < \frac{1}{2}) \text{ and } (Y < \frac{1}{2})) \cup ((X > \frac{1}{2}) \text{ and } (Y > \frac{1}{2}))\big)$. Therefore $A \cap B \cap C = \phi$ and $P(A \cap B \cap C) = 0$, whereas $P(A)P(B)P(C) \neq 0$. A, B and C are not independent.

d)

$$
\begin{aligned}
P((A \cap B^c) \cup (A^c \cap B)) &= P(A) + P(B) - 2P(AB) \\
&= \frac{1}{2} + \frac{1}{2} - 2.\frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

Problem 2:

a)
$A = AB \cup AB^c$, where AB and AB^c are mutually exclusive. Therefore

$$
\begin{aligned}
P(A) &= P(AB) + P(AB^c) \\
&= P(A|B)P(B) + P(A|B^c)P(B^c)
\end{aligned}
$$

b)
i)

$$
P(\text{Second ball is good}|\text{First ball is bad}) = \frac{m}{M-1}
$$

ii)

$$
\begin{aligned}
&P(\text{Second ball is good}) \\
&= P(\text{Second ball is good}|\text{First ball is bad})P(\text{First ball is bad}) \\
&\quad + \text{Second ball is good}|\text{First ball is good}P(\text{First ball is good}) \\
&= \frac{m}{M-1}\frac{M-m}{M} + \frac{m-1}{M-1}\frac{m}{M} \\
&= \frac{m}{M}
\end{aligned}
$$

Problem 3:

a)

$$
\begin{aligned}
P(\cup_{i=1}^n A_i^c) &\leq \sum_{i=1}^n P(A_i^c) \\
\implies P(\cap_{i=1}^n A_i)^c &\leq \sum_{i=1}^n P(A_i^c) \\
\implies 1 - P(\cap_{i=1}^n A_i) &\leq \sum_{i=1}^n P(A_i^c) \\
\implies P(\cap_{i=1}^n A_i) &\geq 1 - \sum_{i=1}^n P(A_i^c)
\end{aligned}
$$

b)

$$P(\cup_{i=1}^{n} A_i^c) = \sum_{i=1}^{n} P(A_i^c) - \sum_{i<j} P(A_i^c \cap A_j^c) + \cdots + (-1)^{n+1} P(A_1^c A_2^c \cdots A_n^c)$$
$$\Rightarrow 1 - P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} (1 - P(A_i)) - \sum_{i<j} (1 - P(A_i \cup A_j)) + \cdots + (-1)^{n+1} (1 - P(A_1 \cup A_2 \cup \cdots \cup A_n))$$
$$\Rightarrow 1 - P(\cup_{i=1}^{n} A_i) = \left(\binom{n}{1} - \binom{n}{2} + \cdots (-1)^{n+1} \binom{n}{n} \right)$$
$$- \left(\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \cdots + (-1)^{n+1} P(A_1 \cup A_2 \cup \cdots \cup A_n) \right)$$
$$\text{Now } (1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 \cdots \binom{n}{n} x^n$$

Putting $x = -1$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} x^2 \cdots (-1)^n \binom{n}{n}$$
$$\Rightarrow \left(\binom{n}{1} - \binom{n}{2} + \cdots (-1)^{n+1} \binom{n}{n} \right) = \binom{n}{0} = 1$$

Therefore

$$1 - P(\cup_{i=1}^{n} A_i) = 1 - \left(\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \cdots + (-1)^{n+1} P(A_1 \cup A_2 \cup \cdots \cup A_n) \right)$$
$$\Rightarrow P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \cdots + (-1)^{n+1} P(A_1 \cup A_2 \cup \cdots \cup A_n)$$

c)

$$\begin{aligned}
&\text{Right hand side} \\
&= P(A|BC)P(B|C) + P(A|B^cC)P(B^cC) \\
&= \frac{P(ABC)}{P(BC)} \frac{P(BC)}{P(C)} + \frac{P(AB^cC)}{P(B^cC)} \frac{P(B^cC)}{P(C)} \\
&= \frac{P(ABC)}{P(C)} + \frac{P(AB^cC)}{P(C)} \\
&= \frac{P(ABC) + P(AB^cC)}{P(C)} \\
&= \frac{P(AC)}{C} \\
&= P(A|C)
\end{aligned}$$

Problem 4:

a)

A is independent of B and A is independent of C. Therefore $P(AB) = P(A)P(B)$and $P(BC) =$

$P(B)P(C)$

$$\begin{aligned}
& A \text{ is independent of } (B\cup C)\\
& \Leftrightarrow P(A\cap(B\cup C)) = P(A)P(B\cup C)\\
& \Leftrightarrow P(A\cap(B\cup C)) = P(A)(P(B)+P(C)-P(B\cap C))\\
& \Leftrightarrow P(AB\cup AC) = P(A)P(B)+P(A)P(C)-P(A)P(B\cap C)\\
& \Leftrightarrow P(AB)+P(AC)-P(AB\cap AC) = P(AB)+P(AC)-P(A)P(B\cap C)\\
& \Leftrightarrow P(ABC) = P(A)P(BC)
\end{aligned}$$

Therefore A is independent of $B\cup C \Leftrightarrow A$ is independent of $B\cap C$

b)

a)

$$\begin{aligned}
& P(\text{Coin is fair}|\text{it gives a head})\\
&= \frac{P(\text{Coin is fair} \cap \text{It gives a head})}{P(\text{Coin gives a head})}\\
&= \frac{P(\text{It gives head}|\text{ Coin is fair})P(\text{Coin is fair})}{P(\text{It gives head}|\text{ Coin is fair})P(\text{Coin is fair}) + P(\text{It gives head}|\text{ Coin is biased})P(\text{Coin is biased})}\\
&= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}}\\
&= \frac{3}{5}
\end{aligned}$$

b)

$$\begin{aligned}
& P(\text{Both are biased}|\text{Both fall heads})\\
&= \frac{P(\text{Both fall heads}|\text{Both are biased})P(\text{Both biased})}{P(\text{Both fall heads})}\\
&= \frac{\frac{1}{9}\cdot\frac{\binom{n}{2}}{\binom{2n}{2}}}{P(\text{Both fall heads})}
\end{aligned}$$

Denominator of the above expression

$$\begin{aligned}
&= P(\text{Both fall Heads})\\
&= P(\text{Both heads}|\text{Both biased})P(\text{Both biased}) + P(\text{Both heads}|\text{Both fair})P(\text{Both fair})\\
&\quad + P(\text{Both heads}|\text{one fair one biased})P(\text{one fair one biased})\\
&= \frac{1}{9}\frac{\binom{n}{2}}{\binom{2n}{2}} + \frac{1}{4}\frac{\binom{n}{2}}{\binom{2n}{2}} + \frac{1}{6}\frac{\binom{n}{1}\binom{n}{1}}{\binom{2n}{2}}
\end{aligned}$$

Therefore

$$P(\text{Both are biased}|\text{Both fall heads}) = \frac{4n-4}{25n-13}$$