Solutions to the Midterm Exam of Winter 2001

1. (a)

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(EF^c) \\ &= P(E) + P(F) - P(E)P(F^c) \\ &= .4 + .5 - (.4)(.5) = .7 \end{aligned}$$

(b) Note that $EG = \emptyset$. Then

$$P(E(F \cup G)) = P(EF \cup EG) = P(EF) + P(EG) - P(EFG) = P(EF) = .2$$

(c) Since E and F^c are independent, $P(E|F^c) = P(E) = .4$

(d) Assume that the sample space is finite having equally likely outcomes. Then

$$P(S) = 1 \geq P(E \cup G) = P(E) + P(G) - P(EG) = .4 + .6 - P(EG) = 1 + P(EG)$$

This derives that $P(EG) = 0$. Since S is a finite set with equally likely outcomes, then $EG = \emptyset$. This implies that $S = E \cup G$ where E and G are mutually exclusive. Thus $E = G^c$. Therefore

$$P(F) = P(F(G \cup G^c)) = P(FG \cup FE) = P(FG) + P(FE)$$

$\Longrightarrow$

$$P(FG) = P(F) - P(FE) = P(F) - P(F)P(E) = .5 - (.5)(.4) = .3$$

(Note that if S is not finite with equally likely outcomes, then EG may not be an empty set.)

2. **(i)** (4 marks) Let H_1 be the event that the first hunter hits, H_2 the second hunter hits, and O the deer is hit by exactly one hunter. Then the desired probability is

$$P(H_2|O) = P(O \cup H_2)/P(O) \tag{1}$$

(ii) (5 marks) Assume that they are independent to hit. Then

$$P(O) = P(H_1H_2^c)+P(H_1^cH_2) = P(H_1)P(H_2^c)+P(H_1^c)P(H_2) = (.3)(.4)+(.7)(.6) = .54$$

and

$$P(OH_2) = P(H_1^c)P(H_2) = (.7)(.6) = .42$$

(iii) (1 mark) Substituting these into (1),

$$P(H_2|O) = .42/.54 = 7/9 = .7778$$

3. Assume that components will function independently. Let X be the number of components that are functional. Then X can be considered as a binomial random variable.
(a)
the p.m.f of X :

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, \cdots, n$$

$$E[X] = np \text{ and } Var(X) = np(1-p)$$

$$E[2X+3] = 2np + 3 \text{ and } Var(2X+3) = 4np(1-p)$$

(b) (3 marks)

$$\begin{aligned} q &= P\{X > 3\} \\ &= P\{X = 4\} + P\{X = 5\} + P\{X = 6\} \\ &= 15(2/3)^4(1/3)^2 + 6(2/3)^5(1/3) + (2/3)^6 \\ &= 2^4 \times 31/3^6 = .6804 \end{aligned}$$

The p.m.f of Y: (2 marks)

$$P\{Y = 1\} = q = .6804 \text{ and } P\{Y = 0\} = 1 - q = .3196$$

4. (a)
(2.5 marks) Let X be the number of database queries by a computer in a 10-second interval. Then we can consider X being a Poisson random variable with parameter $\lambda = 5$.

(2.5 marks) The desire probabilities can be computed as follows.

$$P\{X = 0\} = e^{-5} = 0.0067$$

$$P\{X \geq 2\} = 1 - P\{X < 2\} = 1 - e^{-5} - 5e^{-5} = .9596$$

(b)
Method I
(2.5 marks) Assume that each people randomly gets off on each floor. So the probability of each person gets off on a particular floor is 1/8. We also assume that ten person get off on a particular floor independently.

(2.5 marks) So the number of person getting off on the fifth floor can be considered as a binomial with $n = 10$ and $p = 1/8$. Thus the desired probability is

$$P\{X = 1\} = 10(1/8)(7/8)^9 = 0.3758$$

Method II
There are 8^{10} outcomes that could occur for 10 person getting off, at random, at one of eight floors. (Here we can consider that the urns are the floors and the balls. So this is the number of distributing 10 distinct balls into 8 different urns.) Let E be the event that there is exactly one person getting off on the fifth floor. There are 10 choices for this particular person. In the remaining 7 floors and 9 people, there are 7^9 outcomes that could occur. That's

$$|E| = 10 \cdot 7^9$$

Thus

$$P(E) = \frac{10 \cdot 7^9}{8^{10}} = 0.3758$$