University of Waterloo
Department of Electrical and Computer Engineering

ECE 316 – Probability and Random Processes
Midterm Examination

Thursday, June 19, 2014, 16:30-17:50,
QNC-1502: 20292904 to 20421903
MC-4021: 20422397 to 20446988.

Instructors: Professor M.O. Damen
Time allowed: 80 minutes.
NO AIDS ALLOWED (Some mathematical formulas are given).
Closed book and close notes examination.
Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**
The marking scheme is shown besides the questions and [30] constitutes full mark.
Note that the questions are not of equal difficulty.

Question 1 (6 marks).

(a) [2] Show that if E and F are mutually exclusive, and if $P(E) > 0$ and $P(F) > 0$, then E and F are not independent.

(b) [4] Show that if E and F are independent, then E and F^c are independent and E^c, and F^c are independent.

Question 2 (6 marks) A standard deck of 52 cards consists in 4 suits (spades, clubs, diamonds and hearts) of 13 ranks each $(1, \ldots, 10$, Jack, Queen, King).

(a) [3] If we sample (without replacement) two cards at random out of a well-shuffled deck of 52 cards, what is the probability of having two aces if we know that one of the two cards is the ace of spade (simplify your answer)?

(b) [3] In a 5-cards poker game, a full house occurs when one has three cards of one rank and two cards of another (e.g., three 10's and two kings). Assuming a well-shuffled deck of 52 cards, what is the probability of having a full house?

Question 3 (6 marks) In a town, we have three cars manufacturers that produce all the cars bought in that town: Peter, James, and Ford. Of all the cars manufactured in the town, Peter makes 70%, James 20% and Ford 10%. If 10% of Peter cars are defective, 20% of James cars are defective and 60% of Ford cars are defective:

(a) [3] What is the probability that if you buy a car in that town, it is defective?

(b) [3] Given that your bought car is defective, what is the probability that it was made at Peter's?

Question 4 (6 marks) Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Bin}(n, p)$.

(a) [3] Evaluate $E[3X + 2Y]$ (simplify your answer).

(b) [3] Evaluate $E[X! + e^X]$ (simplify your answer). What are the conditions on λ to have a finite expectation?

Question 5 (6 marks) A transmitter in a communication system consists of 5 antenna arranged in a line. The transmitter will be functional if no two consecutive antennas are defective. Calculate the probability that the transmitter is functional if

(a) [2] Exactly two of the five antennas are defective (assuming all configurations are equally likely).

(b) [4] Each antenna is defective with probability $\frac{1}{2}$, independent of each other.

Useful Formulas

1. Combinatorics:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

2. The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

3. DeMorgan's laws:

$$\left(\bigcup_{j=1}^{n} E_j \right)^c = \bigcap_{j=1}^{n} E_j^c$$

and

$$\left(\bigcap_{j=1}^{n} E_j \right)^c = \bigcup_{j=1}^{n} E_j^c$$

4. Inclusion-exclusion identity (with the notation: $EF = E \cap F$):

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{i=1}^{n} P(E_j) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) + \ldots + (-1)^{n+1} P(E_1 E_2 \ldots E_n)$$

5. Conditional probability: if $P(F) > 0$, then

$$P(E|F) = \frac{P(EF)}{P(F)}$$

6. Multiplication rule:

$$P(E_1 E_2 \ldots E_n) = P(E_1) P(E_2|E_1) P(E_3|E_1 E_2) \ldots P(E_n|E_1 E_2 \ldots E_{n-1})$$

7. Law of total probabilities:
If $F_1, F_2, \ldots, F_n$ form a disjoint partition of the sample space, that is, if $\cup_{i=1}^{n} F_i = \mathcal{S}$ and $F_i F_j = \phi, \forall i \neq j$, then for any event E, one has

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \ldots + P(E|F_n)P(F_n)$$

8. Bayes's formula:

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{P(E)}$$

9. Independent events: The events $E_1, \dots E_n$ are said to be independent if, for every subset of r events, $E_{i_1}, E_{i_2}, \dots, E_{i_r}$ one has

$$P(E_{i_1}E_{i_2}\dots E_{i_r}) = P(E_{i_1})P(E_{i_2})\dots P(E_{i_r})$$

10. Discrete random variables:
If $X \sim \text{Bern}(p)$ then $P(X = 1) = p$ and $P(X = 0) = 1 - p$.
If $Y \sim \text{Bin}(n, p)$ then $P(Y = k) = \binom{n}{k}p^k(1-p)^{n-k}$, $k = 0, 1, \dots, n$.
If $Z \sim \text{Pois}(\lambda)$ then $P(Z = k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, \dots$
The cumulative distribution function (CDF) of X is

$$F(x) = P(X \le x)$$

The expectation of a discrete R.V. X is

$$E[X] = \sum_{p(x)>0} xp(x)$$

and

$$E[g(X)] = \sum_{p(x)>0} g(x)p(x)$$

The variance of X is

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

The standard deviation of X is

$$\sigma = \sqrt{\text{Var}(X)}$$