University of Waterloo
Department of Electrical and Computer Engineering

E&CE-316 – Introduction to Probability Theory

Midterm Examination

June 9, 2006

Instructors: A. K. Khandani
Time allowed: 1.5 hours.
Closed book. One sheet of $8\frac{1}{2} \times 11$ review sheet (one side) allowed.
Answer all three questions. The parts with * sign are more difficult.
Questions and parts within questions are of equal value.

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1. Assume that 5 items are in an urn, numbered 1,2,3,4,5. Suppose that 2 items are white and 3 items are black.

1.1 (mark=2.5) Compute the probability that in a draw of 2 items (without replacement) we obtain 2 white items.

Solution: $\frac{\binom{2}{2}}{\binom{5}{2}} = 0.1.$

1.2 (mark=2.5) Compute the probability that in a draw of 2 items (without replacement) we obtain 2 black items.

Solution: $\frac{\binom{3}{2}}{\binom{5}{2}} = 0.3.$

1.3 (mark=2.5) Compute the probability that in a draw of 2 items (without replacement) we obtain one white and one black item.

Solution: $\frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = 0.6.$

1.4 (mark=2.5) Compute the probability that in a draw of 2 items (with replacement) we obtain 2 white items.

Solution: $(\frac{2}{5})^2.$

2. In a factory, units are manufactured by machines H_1,H_2,H_3 in the proportions $25 : 35 : 40$. The percentages $5\%, 4\%$ and 2%, respectively, of the manufactured units are defective. The units are mixed and sent to the customers.

2.1 (mark=2.5) Find the probability that a randomly chosen unit is defective.

Solution: Let us define D as the event that the unit is defective. Using Bayes's Theorem we have:
$p(D) = p(D/H_1)p(H_1) + p(D/H_2)p(H_2) + p(D/H_3)p(H_3) = 0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4 = 0.0345.$

2.2 (mark=2.5) Suppose that a customer discovers that a certain unit is defective. What is the probability that it has been manufactured by machine H_1?

Solution: $p(H_1/D) = \frac{p(H_1 D)}{p(D)} = \frac{p(H_1)p(D/H_1)}{p(D)} = \frac{0.25 \times 0.05}{0.0345} = 0.362.$

2.3 (mark=2.5) Suppose that a customer buys units until he/she discovers a defective item and then stops buying that product. What is the probability that the customer buys more than 5 items before discovering a defective item.

Solution: Let us define X as the random variable, denoting the number of items the customer buys, before discovering a defective item. Obviously, X has Geometric distribution with parameter $p = 1 - p(D) = 0.9655$. We have:
$p(X > 5) = \sum_{k=6}^{\infty} p^k(1-p) = p^6 = 0.81.$

2.4 (mark=2.5) *Suppose that a customer discovers that a certain unit is defective and we know that it is not produced by H_2, then what is the probability that it has been manufactured by machine H_1?

Solution: This probability can be expressed as $p(H_1/D, H_2^c)$. We have
$p(H_1|D, H_2^c) = \frac{p(H_1 H_2^c D)}{p(DH_2^c)} = \frac{p(H_1 D)}{p(DH_2^c)} = \frac{p(H_1 D)}{p(D(H_1 \bigcup H_3))} = \frac{p(H_1 D)}{p(DH_1 \bigcup DH_3)} = \frac{p(H_1 D)}{p(H_1 D) + p(H_3 D)} = \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.4 \times 0.02} = 0.61.$

3. We know that there are $\binom{n+r-1}{r-1}$ distinct integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying:

$$x_1 + x_2 + \cdots + x_r = n \qquad x_i \geq 0, i = 1, \ldots, r$$

3.1 (mark=5) What is the number of distinct integer-valued vectors $(x_1, x_2, \ldots, x_5)$ satisfying:

$$x_1 + x_2 + \cdots + x_5 = 15 \qquad x_i \geq 2, i = 1, \ldots, 5$$

Solution: Defining $y_i = x_i - 2$, the above equation can be re-written as follows:

$$y_1 + y_2 + \cdots + y_5 = 5 \qquad y_i \geq 0, i = 1, \ldots, 5.$$

Hence, the number of solutions to the primary equation is equal to the number of solutions to the above equation, which is $\binom{5+5-1}{5-1} = \binom{9}{4}$.

3.2 (mark=5) *What is the number of distinct integer-valued vectors $(x_1, x_2, \ldots, x_5)$ satisfying:

$$x_1 + x_2 + \cdots + x_5 = 13 \qquad 4 > x_i > 1, i = 1, \ldots, 5$$

Solution: Again, defining $y_i = x_i - 2$, the above equation can be written as

$$y_1 + y_2 + \cdots + y_5 = 3 \qquad\qquad 2 > y_i > -1, i = 1, \ldots, 5.$$

or equivalently,

$$y_1 + y_2 + \cdots + y_5 = 3 \qquad\qquad y_i = 0 \, or \, 1, \ \ , i = 1, \ldots, 5.$$

Therefore, the problem is simplified to finding the total number of ways that 3 ones and 2 zeros can be put in 5 spots, which is $\binom{5}{3}$.