

E&CE 316: Introduction to Probability Theory
Midterm Examination

Saturday, February 27, 1999
9:00am to 10:30am, MC 2065 & 2066

Instructor: Y. C. Yoon

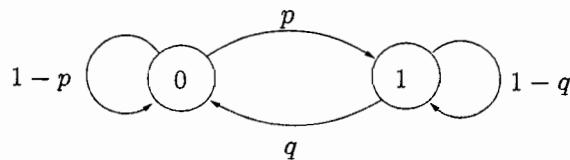
Time allowed: $1\frac{1}{2}$ hours

Closed book examination. Hand calculators are allowed. A formula sheet is attached on the final page.

Attempt all the questions and justify each of your answers clearly.

The marking scheme is shown on the margins. [50] constitutes full marks.

- [5] 1. A data source generates a sequence of four bits (X_1, X_2, X_3, X_4) where $X_i \in \{0, 1\}$ and $i \in \{1, 2, 3, 4\}$. Find the probability of the sequence $(0, 1, 1, 0)$ occurring for the following two situations.
- (a) Assume X_i is a Bernoulli random variable such that $\text{Prob}(X_i = 1) = a$ and [2]
that the four random variables are independent.
- (b) Assume that the sequence starts off with $X_1 = 0$ and obeys the Markov [3]
property with the following state transition diagram



such that $P(X_{j+1} = 1 | X_j = 0) = p$, $P(X_{j+1} = 0 | X_j = 0) = 1 - p$,
 $P(X_{j+1} = 1 | X_j = 1) = 1 - q$ and $P(X_{j+1} = 0 | X_j = 1) = q$ for $j \in 1, 2, 3$.

- [10] 2. (a) In a card game, two cards are randomly selected from a full deck of 52 [4]
cards. What is the probability that they form a blackjack? A blackjack
occurs when one card is an ace and the other is a ten, jack, queen or king.
- (b) Now suppose that two people are playing the same game where each person [6]
receives two cards from the same well-shuffled deck of 52 cards. What is the
probability that neither of them is dealt a blackjack?

- [10]
3. A good hockey team from the east coast plays, at random, two-thirds of their games at home and the remaining one-third on the road. Based on their record at home, the probability of their losing, tying and winning are $\frac{1}{6}$, $\frac{1}{6}$ and $\frac{2}{3}$, respectively. In contrast, based on their record on the road, the probability of their losing, tying and winning are $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$, respectively.
 - (a) In a game they lose, find the conditional probability that they played at home. [4]
 - (b) Is the event of their losing a game independent of their playing at home? Explain why using probabilistic arguments. [3]
 - (c) Find the probability of the event that they play a game at home and either win or tie. [3]
- [10]
4. Given that the length X in minutes of cellular phone conversations during the weekday is a random variable with the probability density function

$$f_X(x) = \begin{cases} cxe^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
 Find:
 - (a) the value of c , [2]
 - (b) the mean length of the conversations, [2]
 - (c) the value of $E[(X + 5)^2]$ and [3]
 - (d) the cumulative distribution function $F_X(x)$. [3]
- [15]
5. Let X be a binomial random variable with the parameters (n, p) .
 - (a) Given n and k where $k \in \{0, 1, \dots, n\}$, what value of p maximizes $P(X = k)$? [4]
 - (b) The result of part (a) can be used to estimate p when a binomial (n, p) random variable is observed to equal k (maximum likelihood estimation). In 100 tosses of a biased coin, 73 tosses come up heads. What is the most likely value of p (the probability of a head)? [2]
 - (c) Given n and p , what is (or are) the most likely value(s) of X ? [7]
 - (d) Given $(n, p) = (3, 0.5)$, what is (or are) the most likely value(s) of X ? [2]

Useful Formula

1. Number of ways to permute k objects from n distinct objects with replacement is n^k .

2. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Number of ways to choose k objects from n distinct objects with replacement is

$$\binom{n+k-1}{k}.$$

5. “Union” identity for two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

6. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

7. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

8. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.

9. Bayes' theorem: $P(A_j|B) = P(B|A_j)P(A_j) / \left[\sum_{i=1}^n P(B|A_i)P(A_i) \right]$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.

10. Binomial distribution: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \dots, n$; $E[X] = np$.

11. Poisson distribution with parameter $\alpha > 0$: $p_X(j) = e^{-\alpha} \frac{\alpha^j}{j!}$, $j = 0, 1, 2, \dots$; $E[X] = \alpha$.

12. Geometric distribution: $p_X(j) = (1-p)^{j-1}p$, $j = 1, 2, \dots$; $E[X] = 1/p$.

13. $\int_0^\infty x^n e^{-x} dx = n!$

14. $\int x e^{ax} dx = e^{ax}(ax - 1)/a^2$

University of Waterloo
Department of Electrical and Computer Engineering



E&CE 316: Probability Theory and Random Processes

Midterm Examination, June 15, 2000

Instructor: Michael Cheung
Time Allowed: 90 minutes
Total Marks: 40
Aids: Non-programmable calculator

Justify your answers

Question 1 [10]

- (a) Bob has 25 packages to send out. In order to evaluate the service of different couriers, he decides to send them through FedEx, UPS, DHL and Purolator. How many different shipping arrangements are possible, if *all* four couriers are used? [2]
- (b) The old Ontario licence plates are made up of i) 3 letters followed by 3 numbers (in the old old days), then ii) 3 numbers followed by 3 letters. How many different licence plates could have been issued with these two numbering schemes? [2]
- (c) Show that
 - i) $P(E_1 E_2 E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2)$, [2]
 - ii) if events E and F are independent, then so are E and F^c , [2]
 - iii) $\text{Var}(2X + 5) = 4\text{Var}(X)$. [2]

Question 2 [10]

- (a) X is binomially distributed with mean 1 and variance 0.8. Find the probability that $X = 0$. [2]
- (b) Find the variance of the Poisson random variable Y , if it is three times as likely that $Y = 4$ than $Y = 2$. [2]
- (c) The random variable Z has pmf of the form

$$P(Z = j) = ap^j, j = 0, 1, 2, \dots$$

- i) For what value of a is this a valid distribution? [2]
- ii) Is this a geometric distribution? [1]
- iii) If $P(Z \geq 4) = 1/256$, find the cdf $F(2)$. [3]

Question 3 [10]

Bob is a Blue Jays fan. He has noticed that the Jays often score many runs in the first inning.

- (a) How would Bob find out the probability that the Jays score more than 3 runs in the first inning? [2]
- (b) How would Bob find out the expected number of runs that the Jays score in the first inning? [2]
- (c) How would Bob find out the variance of the number of runs that the Jays score in the first inning? [2]
- (d) What information does the variance tell? [2]
- (e) What can you say about the Jays' first inning scoring from (b) and (c) together? [2]

Question 4 [10]

In an urn, there are n balls, of which m are red and $n - m$ are blue. All colour composition of the balls are equally likely, i.e., m can be any integer value between 0 and n . In a random experiment, k balls are drawn from the urn successively, *with replacement*.

- (a) Let X be the number of red balls in the urn. What is the pmf of X ? [3]
- (b) Suppose the k samples are all red balls. What is the probability that the urn contains only red balls? [7]

Bonus [2]

A fair coin is tossed 30 times. For $n = 1, 2, \dots, 20$, what is the conditional probability that exactly $10 + n$ heads appear, given that the first 10 tosses resulted in heads?

Useful Formula

1. Number of ways to permute k objects from n distinct objects with replacement is n^k .
2. Number of ways to permute k objects from n distinct objects without replacement is

$$\frac{n!}{(n-k)!}.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Number of ways to divide n indistinguishable objects into k groups where the size of the group can be larger than or equal to zero is

$$\binom{n+k-1}{k-1} \text{ if empty group is allowed; } \quad \binom{n-1}{k-1} \text{ if empty group is not allowed.}$$

5. "Union" identity for two events: $P(A \cup B) = P(A) + P(B) - P(AB)$.
6. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

7. Events A and B are independent if $P(AB) = P(A)P(B)$.
8. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where the events A_i , $i = 1, 2, \dots, n$, partition the sample space into n disjoint regions.
9. Bayes' theorem:

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

where the events A_i , $i = 1, 2, \dots, n$, partition the sample space into n disjoint regions.

10. Binomial distribution: $p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$, $k = 0, 1, 2, \dots, n$. $E[X] = np$, $\text{Var}(X) = np(1-p)$.
11. Poisson distribution with parameter $\lambda > 0$: $p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \dots$. $E[X] = \text{Var}(X) = \lambda$.
12. Geometric distribution: $p_X(k) = (1-p)^{k-1}p$, $k = 1, 2, \dots$. $E[X] = 1/p$, $\text{Var}(X) = (1-p)/p^2$.
13. Negative binomial distribution: $p_X(k) = \binom{n-1}{k-1}p^k(1-p)^{n-k}$, $n = k, k+1, \dots$. $E[X] = r/p$, $\text{Var}(X) = r(1-p)/p^2$.
14. $\sum_{i=1}^m i = \frac{m(m+1)}{2}$, $\sum_{i=0}^m x^i = \frac{1-x^{m+1}}{1-x}$, $x \neq 1$, $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.



University of Waterloo Mid-Term Examination

Spring **2001**
(Term) (Year)

Student Name

Student ID Number

Course Abbreviation and Number E&CE 316

Course Title Probability Theory and Random Processes

Instructor G. Gong

Date of Exam 6/14/01

Time Period:	4:30–6:30 pm	Start Time:	16:30:00	End Time:	18:00:00
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Duration of Exam 1.5 hours

Number of Exam Pages 3 pages
(including this cover sheet)

Exam Type Closed book

Additional Materials Allowed One 8.5 by 11 review sheet included

Marking Scheme: ANSWER ALL THE QUESTIONS. QUESTIONS OF EQUAL MARK. 40 CONSTITUTES FULL MARK.

Question 1: (10/40) A pair of fair dice are rolled. Let:

- E be the event that the sum of the two numbers shown on the dice is 7;
- F be the event that the second die lands on a higher value than does the first;
- G be the event that the number shown on the both dice are equal.

Compute:

- (a) $(3/10) P(E)$, $P(F)$ and $P(G)$;
- (b) $(3/10) P(E \cup F \cup G)$;
- (c) $(4/10) P(E|F)$ and $P(E|G)$; are E and G independent?

Question 2: (10/40) A large lot of circuits is known to contain a fraction θ defective. Let X be a random variable representing the number of circuits that are defective in the first n inspected circuits.

- (a) Find the probability mass function of X .
- (b) Find the expectation and the variance of X .
- (c) Repeat questions (a) and (b) for $n = 100$ and $\theta = 0.011$.
- (d) For $n = 200$ and $\theta = 0.01$, compute $E[X]$, $Var(X)$, $E[20X + 33]$, and $Var(12X + 99)$.

Question 3: (10/40) The number of buses that arrive at a bus stop in T minutes is a Poisson random variable, X , with expected value $T/10$.

- (a) What is the p.m.f. of X , the number of buses that arrive in T minutes? Repeat it for $T = 20$.
- (b) What is the probability that in a two minute interval, three buses will arrive?
- (c) What is the probability that no buses arriving in a 10-minute interval?
- (d) How much time should you allow so that with probability 0.99 at least one bus arrives?

Question 4: (10/40) Solve the following problems.

- (a) (5/10) Independent trials are performed. Each has a success with probability .33 and a failure with probability .67. Let X be a random variable representing the number of trials to be performed to obtain the first failure.
 - (i) Find the p.m.f. of X and $E[X]$.
 - (ii) Compute $F(0)$, $F(2)$, and $P\{X \leq 3\}$ where $F(b)$ is the c.d.f. of X .
- (b) (5/10) 24 men randomly select hats from a set of consisting of one hat from each person. Assume that each of the men selects one hat and checks if it is his own hat, then puts it back. What is the probability that
 - (i) none of the men selects his own hat;
 - (ii) exactly three of the men select their own hats?

1. The number of permutations of r objects from n distinct objects is

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}, \quad r \leq n$$

2. The number of combinations of selecting r objects out of n is given by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$
3. The number of ways to distribute n distinct balls into r different urns is r^n .
4. The number of ways to distribute n identical balls into r different urns without empty urns is $\binom{n-1}{r-1}$. If empty urns are allowed, then this number is $\binom{n+r-1}{r-1}$.
5. Probability of union of events:

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$$

6. If $P(F) > 0$, then $P(E|F)$, the conditional probability of event E given that F has occurred, is

$$P(E|F) = \frac{P(EF)}{P(F)}$$

7. Total probability:

$$P(E) = \sum_{i=1}^n P(E|F_i)P(F_i)$$

where the events $F_j, j = 1, 2, \dots, n$, are a partition of the sample space.

8. Bayes' formula,

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)}$$

where the events $F_j, j = 1, 2, \dots, n$, are a partition of the sample space.

9. Two events E and F are independent if $P(EF) = P(E)P(F)$.
10. $\sum_{i=1}^n = \frac{n(n+1)}{2}$, $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, $\sum_{n=0}^{\infty} q^n = 1/(1-q)$, $0 < q < 1$.

	Random variable X	Parameter	the p.m.f. $p(k)$	Expectation $E[X]$	Variance $Var(X)$
11.	Binomial	(n, p)	$\binom{n}{k} p^k (1-p)^{n-k}$ $k = 0, 1, \dots, n$	np	$np(1-p)$
	Poisson	λ	$e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$	λ	λ
	Geometric	p	$(1-p)^{k-1} p, k = 1, \dots$	$1/p$	$(1-p)/p^2$

¹ This sheet will be attached with the mid-term exam.

University of Waterloo
Department of Electrical and Computer Engineering

E&CE-316 – Probability Theory and Random Processes
Midterm Examination

February 9, 2002, 9:00pm to 10:30pm, MC 4020, 4021, 4059

Instructor: Y. Dabbagh and G. Gong

Time allowed: 1.5 hours.

Closed book examination. One $8\frac{1}{2} \times 11$ review sheet allowed.

Answer all the questions. Questions of equal mark. [50] constitutes full mark.

Note that the questions are not of equal difficulty.

Question 1. (10/50) In an experiment, E , F and G are three events with probabilities $P(E) = .5$, $P(F) = .2$ and $P(G) = .3$. Assume that E and G are mutually exclusive, $P(EF) = .1$ and $P(FG) = .2$. Find the following probabilities.

- (a) (2.5/10) Find $P(E \cup F \cup G)$
- (b) (2.5/10) Prove that $P(E^c) = 1 - P(E)$.
- (c) (2.5/10) Compute $P(E \cup G^c)$
- (d) (2.5/10) Compute $P(E|F^c)$. Are E and F^c independent? Explain your reasoning.

Question 2. (10/50) There are two urns, urn A contains 3 white balls and 2 black balls, and urn B contains 5 white balls and 1 black ball. Three balls are withdrawn from exactly one of these two urns randomly, and they all are white. What is the probability that these 3 white balls are drawn from urn B?

Question 3. (10/50) Suppose that a biased coin that lands on head with probability p is flipped n times independently.

- (a) (2.5/10) What is the probability that the heads will land at least $n - 2$ times? Repeat it for $n = 6$ and $p = .25$.
- (b) For each flip, if a head lands, you win \$2, and then you win -\$1 (that is, you lose) if a tail lands. Let Y denote the amount of your winnings. Find $E[X]$ and $Var(Y)$. Repeat it for $n = 10$ and $p = .25$.
- (c) Suppose that you win \$5.00 if an even number (but not equal to zero) of heads results, you win -\$1.00 if an odd number of heads results, and then you win nothing if no heads results. Let Y denote the amount of your winnings. Find the probability mass function of Y if $n = 5$ and $p = .45$.
- (d) If $n = 9$ and $p = .25$, what is the conditional probability that the first 5 flips land on head and the 6th lands on tail given that a total of 6 heads results?

Question 4. (10/50) Digital signature is a way to sign a document electronically. Assume that the expected number of forged digital signatures occurred at some e-banking system in a 4-weeks interval is 3.

- (a) (5/10) What is the probability that there will be at least 1 forgery in the next week?
- (b) (5/10) What is the probability that there will be at most 3 forgeries in the next week?

Explain your reasoning.

Question 5. (10/50) There are n components lined up in a linear arrangement. Let X denote the number of the components that are defective, then $n - X$ is the number of the components that are functional. Assume that all of the defectives and all of the functionals are considered indistinguishable.

- (a) (5/10) If $n = 8$, how many linear orderings are there in which no 2 neighboring components are both defectives for $X = 0, 1, 2, \dots, 8$?
- (b) (5/10) If $n = 8$, what is the probability that no 2 neighboring components are both defectives?

Useful Formulae¹

1. The number of permutations of r objects from n distinct objects is

$$n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}, \quad r \leq n$$

2. The number of combinations of selecting r objects out of n is given by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

3. The number of ways to distribute n distinct balls into r different urns is r^n .

4. The number of ways to distribute n identical balls into r different urns without empty urns is $\binom{n-1}{r-1}$. If empty urns are allowed, then this number is $\binom{n+r-1}{r-1}$.

5. Probability of union of events:

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$$

6. If $P(F) > 0$, then $P(E|F)$, the conditional probability of event E given that F has occurred, is

$$P(E|F) = \frac{P(EF)}{P(F)}$$

7. Total probability:

$$P(E) = \sum_{i=1}^n P(E|F_i)P(F_i)$$

where the events $F_j, j = 1, 2, \dots, n$, are a partition of the sample space.

8. Bayes' formula,

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)}$$

where the events $F_j, j = 1, 2, \dots, n$, are a partition of the sample space.

9. Two events E and F are independent if $P(EF) = P(E)P(F)$.

10. The Binomial distribution with parameter (n, p) : $p(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, \dots, n, E[X] = np, \text{Var}(X) = np(1-p)$.

11. The Poisson distribution with parameter λ : $p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots, E[X] = \lambda, \text{Var}(X) = \lambda$.

12. $\sum_{i=1}^n i = \frac{n(n+1)}{2}, e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.

¹This sheet will be attached with the mid-term exam.

University of Waterloo
Department of Electrical and Computer Engineering
Friday, February 14, 2003
5:30pm to 7:00pm.

ECE316 Midterm Examination

Instructor: W. Zhuang

Time allowed: 1.5 hours.

NO AIDS ALLOWED except a hand calculator (Some useful facts are given on page 2).

Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

The marking scheme is shown in the left margin and [40] constitutes full marks.
You can earn 5 bonus marks!

- [10] 1. If A , B , and C are three events defined on the same sample space, explain why the following assignments of probabilities are impossible:
- [3] (a) $P(A) = 0.65$, $P(AB^c) = 0.6$, $P(AB) = 0.1$;
- [3] (b) $P(A) = P(B) = P(C) = 0.5$, $P(AB) = 0$, $P(ABC) = 0.2$;
- [4] (c) $P(ABC) = 0$, $P(A) = P(B) = 0.4$, $P(C) = 0.5$, $P(A \cup B \cup C) = 0.55$, $P(AC) = P(AB) = 1$.
- [10] 2. A multiple-choice test contains 10 questions, each having 5 possible answers. A student has a probability θ of knowing the answer to any particular question. For a question to which he does not know the answer, he guesses and has a probability $1/5$ of guessing correctly. If each question is worth ten marks and is assigned a mark of 0 or 10, what is his expected mark?
- [10] 3. A point X is picked at random on a rod of length 1. What is the probability that the area of the rectangle, with sides of lengths chosen as the lengths from X to the ends of the rod, exceeds $1/5$?

- [10] 4. A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a “false positive” result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability 0.01, the test result will imply that he or she has the disease.) If 0.5 percent of the population actually has the disease, determine
- [5] (a) the probability that the test result is positive, and
- [5] (b) the probability that a person has the disease, given that the test result is positive.
- [5] 5. **[Bonus]** A friend proposes the following game. She will select at random some number from the interval $[0, 100]$ and offer you that amount in cash. You may accept the money or reject. If you accept it, the game ends; otherwise, she will choose another number at random from the same interval and now offer you that amount. As before, you may either accept this second offer or reject it. If you accept it, the game ends; otherwise, she will make a third and final offer (again choosing a number at random from the interval); you must accept this one. Should you play the game if the cost for playing is \$65, and if you do elect to play, what should your strategy be?

Useful facts:

1. Binomial distribution:

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

2. Poisson distribution with parameter $\lambda(>0)$: $p_X(j) = e^{-\lambda} \lambda^j / j!$, $j = 0, 1, 2, \dots, \infty$.

3. Exponential distribution with parameter $\lambda(>0)$: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

4. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

5. Events A and B are independent if $P(AB) = P(A)P(B)$.

6. “Union” identity:

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

7. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where A_i ’s form a partition of the sample space.

8. Bayes theorem: $P(A_j|B) = P(B|A_j)P(A_j) / [\sum_{i=1}^n P(B|A_i)P(A_i)]$, where A_i ’s form a partition of the sample space.

E&CE 316 – Probability Theory and Random Processes
Department of Electrical and Computer Engineering
University of Waterloo
Instructors: P. Ho and K. T. Wong

Midterm Examination

5:30-7:00pm, June 18, 2003
RCH 302, 305, 307, 309

Time allowed: 1:30 hours.

1. Close book/notes. One formula-sheet provided; no other formula-sheet allowed. No electronic devices (computers, PDA's, cell phones, pagers, etc) allowed, except that a hand-held calculator is optional.
2. All questions need to be answered. Each question is worth 100 points. The maximum point-total in this examination is 300, the minimum is -60.
3. For Q1, no partial credit will be given. **The student MUST write down his/her Q1's version # (along with his/her answers) to receive any credit for Q1.**
4. Full/partial credit given for Q2 and Q3 only with detailed intermediate steps. Correct answers with inadequate details and/or intermediate steps may result in zero credit in Q2 and Q3.
5. All answers must be written only on the examination booklet, not on the question sheets.

Q #1 (100 points maximum, -60 points minimum, written by Professor Wong)

25 points for each correct answer. -15 points for any incorrect answer. Any unanswered part results in 0 point. Any part with more than one answer will receive 0 point.

You MUST write in your exam booklet, next to your Q1 answers, that you have version #8.

(A) An ordinary deck of 52 cards is shuffled and the cards are then turned over one at a time, until an ace appears. Given that the first ace is the 10th card to appear, what is the conditional probability that the 11th card would be the ace of club?

- a. $3 / 204$
- b. $3 / 168$
- c. $1 / 42$
- d. $1 / 51$
- e. $1 / 52$

(B) All workers at a certain company drive to work and park in the company's lot. The company is interested to estimate the average number of workers in a car. Two methods are proposed:

#1: Randomly choose n workers; find out how many people were in the cars in which they were driven, and take the average of the n values.

#2: Randomly choose n cars in the lot; find out how many people were in those cars, and take the average of the n values.

Which method(s) should be used?

- a. Method #1
- b. Method #2
- c. Either one. Methods #1 and #2 are statistical equivalent methods.
- d. Do both #1 and #2 above and take their average.

(C) A pair of unfair dice is rolled. The probability of each die giving a 1 or a 2 or a 3 is $1/9$. The probability of each die giving a 4 or a 5 or a 6 is $2/9$. What is the probability that the first die lands on a lower value than does the second?

- a. $1/4$
- b. $1/3$
- c. $11/27$
- d. $5/12$
- e. $1/2$

(D) A gambler has in his/her pocket a fair coin and a two-headed coin. He/she selects one of the coins at random, with probability $p=2/3$ of selecting the fair coin. Then he/she flips it $n=2$ times. It shows heads all $n=2$ times. What is the probability that it is the two-headed coin?

- a. $2/3$
- b. $5/9$
- c. $1/2$
- d. $4/9$
- e. $1/3$

Q #2 (100 points, written by Professor Ho)

Full/partial credit given only if detailed intermediate steps are shown. Correct answer with inadequate details and/or intermediate steps may result in zero credit. In each part below, you need not simplify the arithmetic in your solution's final-step, if you have no calculator.

Consider a receiver with arrivals of data packet, in which the number of arrivals follows a Poisson distribution with an arrival rate 5 arrivals/sec.

- (a) What is the probability that the number of arrivals is more than 3 but less than 6 during a time interval of 2 seconds? (10 points) (Hint: the PMF of a Poisson random variable is

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \text{ where } \lambda \text{ is the corresponding parameter})$$

- (b) The inter-arrival time, which is the time interval between two consecutive arrivals, follows an exponential random variable with a PDF $f_t(t) = \alpha e^{-\alpha t}$ and a CDF $F_t(t) = 1 - e^{-\alpha t}$. What would be the parameter α in this case? (15 points) What is the probability that the inter-arrival time is less than 0.3 seconds? (15 points)

- (c) If the receiver can make money according to the number of data packets it deals with. The relationship between the revenue and the number of received data packets by a single receiver is $Y = 3 \cdot X + 2$, where Y is the random variable of the revenue made by this receiver within 3 seconds, and X is the random variable of the number of the received data packets within 3 seconds. What is the expectation of Y ? Assume the receiver never fails. (10 points) (Hint: Use $E[Y] = \sum g(x) \cdot P(X = x)$ if we have the function $Y = g(X)$, where $P(X = x)$ is the PMF of the random variable X).

- (d) If the receiver has a life time T following Normal distribution $T \sim N(\mu, \sigma)$, where both $\mu = 10^6$ and $\sigma = 8 \times 10^5$ are in a unit of seconds. What is the probability that the receiver fails to last 1.5×10^6 seconds? (10 points) Illustrate and identify the area standing for the probability in the corresponding Normal curve. (15 points)

- (e) If $\mu = 10^6$ and $\sigma = 2 * 10^5$, what is the probability that the receiver fails to last 1.2×10^6 seconds given that it has lasted 10^6 seconds? (25 points)

Area $\Phi(x)$ under the standard normal curve to the left of x (same table as in Ross' p. 203)

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Q #3 (100 points, written by Professor Wong)

Full/partial credit given only if detailed intermediate steps are shown. Correct answer with inadequate details and/or intermediate steps may result in zero credit.

(A) 60 points:

The random variable X has cumulative distribution function (CDF) $F_X(x)$. What is the cumulative distribution function (CDF) of the random variable $Y = aX + b$, where a and b may each be any real-value number? Express your answer in terms of $F_X(x)$, a and b .

(B) 40 points:

X is a random variable. Is it generally true that $E[1/X] = 1/E[X]$? Why? Is it ever true that $E[1/X] = 1/E[X]$? Why? (A simple “yes” or “no” will receive no credit.)

University of Waterloo
Department of Electrical and Computer Engineering
E&CE-316: Probability Theory and Random Processes
Midterm Examination
February 13, 2004, 5:30pm to 7:00pm, RCH 305, 307, 309

- **Instructor: Ali Abedi**
- **Time allowed: 90 minutes.**
- **Closed book examination. One 8.5×11 review sheet allowed.**
- **Answer all the problems. Each problem has 5 marks.**

Problem 1: A cable company has introduced two new products (wireless service and High speed Internet) to increase its revenue. This action increased the probability of selling a product from 72% to 86%. Surveys showed that although the wireless and Internet are equally popular, but no one purchases both of them at the same time. Assuming that the probability of selling cable with one more product is 34%, what is the probability of selling wireless service?

Problem 2: There are **r** red, **b** blue, and **g** green balls in a box. Taking **k** balls out of the box at random, what is the probability of having at least one ball from each color?

- a) Without replacement.
- b) With replacement.

Problem 3: Only 2% of the Engineers consider graduate studies. After obtaining the graduate degree, 21% end up with an academic job and the rest go for the industry. However, there is still a 1% chance of getting an academic job with out any graduate degree. Selecting a person at random from academia, what is the probability that he/she has a graduate degree?

Problem 4:

- a) Find the number of positive solutions to the following equation:

$$x_1 + x_2 + x_3 + x_4 = 32$$

With these conditions: $x_1 > 5$, $x_2 \geq 3$, $x_3 > 7$

- b) How many non-negative solutions are there?

Problem 5: A train with **n** passengers makes 10 stops before arriving at its final destination, where all of the passengers should get off. Random variable **X** represents the number of passengers who get off the train at final destination. Assuming that a passenger gets off at any of the middle stops with equal probability,

- a) Compute the p.m.f. of **X**.
- b) What is the expected value and variance of **X**?

Problem 6: To transfer a file completely, we need 12000 packets to arrive at a node in the network. Random variable **Z** denotes the number of lost packets during the file transfer. Assuming the probability of packet loss is 10^{-6} , compute exact and approximate p.m.f. of **Z**. What is the probability of no packet loss using the approximate p.m.f.?

E&CE 316 – Probability Theory and Random Processes
Midterm Examination

6:00-7:30pm, June 13, 2005
DWE 3516, 3518, 3522

Instructor: K. T. Wong, Ph.D.

Department of Electrical and Computer Engineering
University of Waterloo

Time allowed: 1:30 hours.

1. Close book/notes. One formula-sheet provided; no other formula-sheet allowed. No electronic devices (computers, PDA's, cell phones, pagers, etc) allowed.
2. All questions need be answered, and only in the examination booklet not on the question sheets.
3. Full/partial credit given only with detailed intermediate steps. **Correct answers with inadequate details and/or intermediate steps may result in zero credit.**

Q #3 (5 points)

A gambler has in his/her pocket a fair coin and a two-headed coin. He/she selects one of the coins at random, with probability $p=1/3$ of selecting the fair coin. Then he/she flips it $n=3$ times. It shows heads all $n=3$ times. What is the probability that it is the two-headed coin?

Solution:

$$\begin{aligned} & P\{2\text{-headed coin} | hhh\} \\ = & \frac{P\{2\text{-headed coin \& } hhh\}}{P\{hhh\}} \\ = & \frac{P\{hhh | 2\text{-headed coin}\} P\{2\text{-headed coin}\}}{P\{hhh | 2\text{-headed coin}\} P\{2\text{-headed coin}\} + P\{hhh | \text{fair coin}\} P\{\text{fair coin}\}} \\ = & \frac{(1)\left(\frac{2}{3}\right)}{(1)\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)} = \frac{2}{2 + \left(\frac{1}{8}\right)} = \frac{16}{17} \end{aligned}$$

Q #4 (4 points)

People enter the library at the average rate of 1 person for every 3 minutes. What is the probability that nobody enters between 12:04:00 and 12:09:00? The answer may be expressed in terms of trigonometric functions or a number being raised to a certain power.

Solution:

$$\lambda=5/3. P\{\text{no one enters between 12:04:00 and 12:09:00}\} = e^{-5/3} (5/3)^0 / 0! = e^{-5/3}$$

University of Waterloo
Department of Electrical and Computer Engineering
Wednesday, Feb. 16, 1994
4:30 to 6:00 p.m.

E&CE-316 Midterm Examination

Instructors: G. Kesidis, 3A Computer Engineering
W. Zhuang, 3A Electrical Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED (A formula sheet is attached to this examination as page 2).

Attempt all 5 questions. **JUSTIFY ALL YOUR ANSWERS.**

FILL IN YOUR NAME AND NUMBER BELOW.

The marking scheme is shown in the left margin and [50] constitutes full marks.

Note that the questions are not of equal difficulty.

NAME: _____

ID NUMBER: _____

Number of ways to permute k objects from n distinct objects without replacement is

$$\frac{n!}{(n-k)!}.$$

Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

$$\text{Binomial Theorem: } (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

For $P(B) \neq 0$, the conditional probability of event A given event B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Theorem: for $P(B) \neq 0$ and $P(A) \neq 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The Law of Total Prob.: If $\bigcup_{i=1}^n A_i = S$ and $A_j \cap A_i = \emptyset$ for all i, j such that $i \neq j$, then for any event B ,

$$P(B) = \sum_{i=1}^n P(B \cap A_i),$$

and (combined with Bayes' theorem) for $P(B) \neq 0$ and $P(A_j) \neq 0$ for all j ,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}.$$

"Union" identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$A_i, i = 1, \dots, n$ are pairwise independent if for any i, j such that $i \neq j$,

$$P(A_i \cap A_j) = P(A_i)P(A_j)$$

$A_i, i = 1, \dots, n$ are mutually independent if for any collection of distinct indices $i_1, i_2, \dots, i_k$ with $2 \leq k \leq n$,

$$P\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k P(A_{i_j})$$

- [10] 1. The ratio X of the deviations from the nominal length and width of silicon chips has the probability density function (pdf)

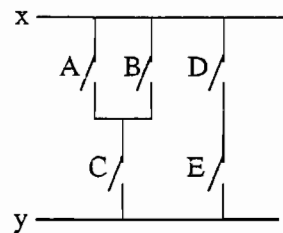
$$f(x) = \frac{C}{1+x^2}, \quad -\infty < x < \infty$$

where C is a constant.

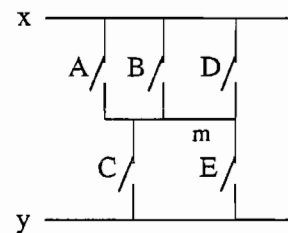
- [1] (a) Find the value of C .
- [3] (b) Find the cumulative distribution function (cdf) $F(x)$ of the random variable X .
- [3] (c) Find the expectation $E(X)$ and variance $V(X)$.
- [3] (d) Find the probability that X^2 lies between $1/3$ and 1 .

Hint: $\frac{d}{dx} \tan^{-1} x = 1/(1+x^2)$.

- [10] 2. The network of switches A, B, C, D and E are connected across the power lines x, y as shown in (a). Each switch has probability $p = 0.4$ of not closing when operated, and each switch functions independently of the other switches. What is the probability that the circuit from x to y will fail to close when all five switches are operated? Will the addition of bus m (as shown in (b)) increase or decrease this probability?

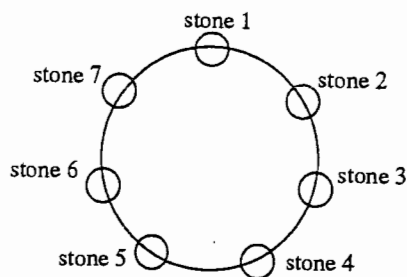


(a)



(b)

- [5] 3. (a) If two fair dice are tossed, what is the smallest number of throws, n , for which the probability of getting at least one double 6 exceeds 0.5?
- [5] (b) If two fair dice are tossed, what is the probability of getting a double 6 given that the numbers on the upsides of both dice are ≥ 4 ?
- [5] 4. (a) An urn contains five white chips, four black chips, and three red chips. Four chips are drawn sequentially and without replacement. What is the probability of obtaining the sequence (white, red, white, black)?
- [5] (b) How many different ways can you arrange seven different colored stones on a necklace (as shown in the following figure)?



- [10] 5. A toy manufacturer buys ball bearings from three different suppliers: 50% of his total order comes from supplier 1, 30 % from supplier 2, and the rest from supplier 3. Past experience has shown that the quality control standards of three suppliers are not all the same. Of the ball bearings produced by supplier 1, 2% are defective, while suppliers 2 and 3 produce defective bearings 3% and 4% of the time, respectively. Given that a ball bearing bought by the toy manufacturer is defective, what is the probability that it came from supplier 1?

University of Waterloo
Department of Electrical and Computer Engineering
Saturday, June 11, 1994
10:00am to 11:30am.
E&CE-316 Midterm Examination

Instructors: W. Zhuang, 3A Electrical Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED (A formula sheet is attached to this examination as page 2).
Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

FILL IN YOUR NAME AND NUMBER BELOW.

The marking scheme is shown in the left margin and [50] constitutes full marks.
You can earn 5 bonus marks!

NAME: _____

ID NUMBER: _____

Useful facts:

I. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

II. Number of ways to permute k objects from n distinct objects with replacement is

$$n^k.$$

III. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

IV. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

V. Bayes' Theorem: Let $\{A_i\}_{i=1}^n$ be a set of events defined over S such that $\bigcup_{i=1}^n A_i = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, and $P(A_i) > 0$ for $i = 1, 2, \dots, n$. For any event B (also defined on S), where $P(B) > 0$,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}.$$

VI. Binomial Distribution: Let p be the probability that event A occurs for a trial. The probability that event A occurs k times in n independent trials ($0 \leq k \leq n$) is

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

VII. "Union" identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

VIII. The sum of a geometric series

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}, \quad \text{if } |r| < 1.$$

1. The random variable X has a cdf given by

$$F(x) = \begin{cases} 0, & -\infty < x < 0 \\ x^2, & 0 \leq x < 1 \\ 1, & 1 \leq x < \infty. \end{cases}$$

- [2] (a) Find pdf of X .
- [2] (b) Find $P(\frac{1}{2} < X \leq \frac{3}{4})$.
- [3] (c) Find the expectation of X .
- [3] (d) Find the variance of X .

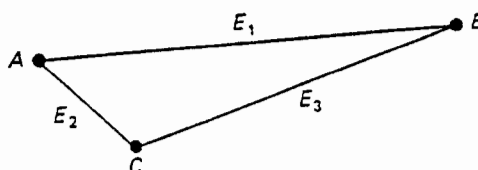
2. Consider a countably infinite sample space consisting all the positive integers except 1. Define

$$P(i) = C\left(\frac{2}{3}\right)^i, \quad i = 2, 3, 4, \dots$$

- [3] (a) Find the value of C .
- [3] (b) What is the probability that the outcome of the experiment will be less than or equal to 4?
- [4] (c) What is the probability that the outcome will be an even number?

- [5] 3. (a) An urn contains six chips numbered 1 through 6. Three are drawn out without replacement. What is the probability of the event "second smallest chip is a 3"?
- [5] (b) Six dice are rolled one time. What is the probability that each of the six dice shows a unique face?
- [5] (c) Ten electronic components are put on a life test. The time T to failure of a component is a random variable that is exponentially distributed with parameter α , that is, the pdf of T is $f_T(t) = \alpha e^{-\alpha t}, t > 0$. Find the probability that at least 8 of the 10 components will still be operative after 150 hours.

- [5] 4. The highways connecting two resort areas at A and B are shown in the following figure. There is a direct route through the mountains and a more-circuitous route going through a third resort area at C in the foothills. Travel between A and B during the winter months is not always possible, the roads sometimes being closed due to snow and ice.



Suppose we Let E_1 , E_2 and E_3 denote the events that highways AB , AC and CB are passable, respectively, and we know from past years that on a typical winter day,

$$P(E_1) = 2/5, \quad P(E_2) = 3/4, \quad P(E_3) = 2/3$$

$$P(E_3|E_2) = 4/5, \quad P(E_1|E_2 \cap E_3) = 1/2$$

What is the probability that a traveler will be able to get from A to B ?

- [10] 5. During a power blackout, 100 persons are arrested on suspicion of looting. Each is given a polygraph test. From the past experience, it is known that the polygraph is 90% reliable when administered to a guilty suspect and 98% reliable when given to someone who is innocent. Suppose that of the 100 persons taken into custody, only 12 were actually involved in any wrongdoing. What is the probability that a suspect is innocent given that the polygraph says he or she is guilty?
- [5] 6. [Bonus] Two gambler, A and B , each choose an integer from 1 to m (inclusive) at random. What is the probability that the two number they pick do not differ by more than n for $0 \leq n \leq m - 1$?

University of Waterloo
Department of Electrical and Computer Engineering
E&CE-316 – Introduction to Probability Theory
Midterm Examination
17 February, 1995

Instructors: I. F. Blake and A. K. Khandani

Time allowed: 1.5 hours.

Closed book examination. One $8\frac{1}{2} \times 11$ review sheet (one side) allowed.

Answer any three questions. If all four questions attempted, best three count.

Questions of equal value. [30] constitutes full marks.

Problem 1: Alice and Bob are among twelve people taking part in a recital. Twelve seats are arranged in a line on the stage and the twelve participants choose their seat at random. What is the probability that:

- (3) 1.1. Alice is in an end seat?
- (3) 1.2. Alice and Bob are both in end seats?
- (2) 1.3. Alice and Bob sit next to each other?
- (2) 1.4. Alice sits some where to the left of Bob (not necessarily adjacent to him)?

Problem 2: Consider a game in which two persons A and B guess a two digit number with the digits belonging to the set $\{1, 2, \dots, K\}$. If the two numbers match in the first place, player A pays X dollars to player B , and if the two numbers match in the second place, player B pays the same amount to player A (if the two numbers match in both places, neither of them pays).

- (3) 2.1. Assume that the two players are allowed to have repeated digits in their numbers. What is the mean and the variance for the gain of each player.
- (4) 2.2. Assume that the two players are not allowed to have repeated digits in their numbers. What is the mean and the variance for the gain of each player.
- (3) 2.3. In a slightly different scenario, the two players select a three digit number and if the numbers match in the first place, A pays X dollars to B , and if they match in the third place, B pays X dollars to A (if the two numbers match in both places, neither of them pays). What is the mean and the variance for the gain of each player in parts (2.1) and (2.2) above for this case.

Problem 3: A Professor sets an examination with one question for two students. Of the two students, she knows that Student 1 has a probability of .3 of answering the question correctly and Student 2 has a probability of .5 of answering the question correctly.

- (4) 3.1. She chooses a paper at random and finds it contains a correct answer. What is the probability it is the paper of Student 1?
- (3) 3.2. What is the probability that exactly one of the students got the answer correct?
- (3) 3.3. Her TA actually marked the papers and tells her that exactly one student got the answer correct. What is the probability that Student 1 answered the question correctly?

Problem 4: A semi-conductor factory produces chips each containing 8 logical units. Due to the impurity in the silicon surface, a logical unit may fail to function properly after production. The pdf for the percentage of impurity in the area occupied by a given logical unit is as shown in Fig. 1. Each logical unit fails if the percentage of impurity in its corresponding area is more than 25%. For the sake of simplicity, we assume that the logical units fail independently of each other.

To increase the production efficiency, the chips are designed to have $K \geq 8$ logical units. In the quality control, those with at least 8 units functioning properly are selected and 8 of their good units are given external connection to the pins.

- (3) 4.1. Compute the probability that a given logical unit fails.
- (3) 4.2. Give an expression for the probability that a given chip contains at least 8 good units.
- (4) 4.3. Assume that the probability of making a mistake in measuring the percentage of impurity in silicon surface (and deciding whether a given logical unit is good or not) is equal to 0.1. What is the probability that a given logical unit passes the quality control, and what is the conditional probability that such unit is defective.

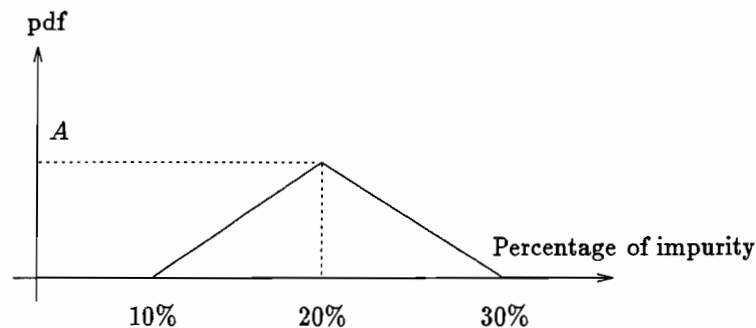


Figure 1: pdf for the percentage of impurity in silicon surface.

University of Waterloo
Department of Electrical and Computer Engineering
E&CE-316 – Introduction to Probability Theory
Midterm Examination
17 February, 1995

Instructors: I. F. Blake and A. K. Khandani

Time allowed: 1.5 hours.

Closed book examination. One $8\frac{1}{2} \times 11$ review sheet (one side) allowed.

Answer any three questions. If all four questions attempted, best three count.

Questions of equal value. [30] constitutes full marks.

Problem 1: Alice and Bob are among twelve people taking part in a recital. Twelve seats are arranged in a line on the stage and the twelve participants choose their seat at random. What is the probability that:

- (3) 1.1. Alice is in an end seat?
- (3) 1.2. Alice and Bob are both in end seats?
- (2) 1.3. Alice and Bob sit next to each other?
- (2) 1.4. Alice sits some where to the left of Bob (not necessarily adjacent to him)?

Problem 2: Consider a game in which two persons A and B guess a two digit number with the digits belonging to the set $\{1, 2, \dots, K\}$. If the two numbers match in the first place, player A pays X dollars to player B , and if the two numbers match in the second place, player B pays the same amount to player A (if the two numbers match in both places, neither of them pays).

- (3) 2.1. Assume that the two players are allowed to have repeated digits in their numbers. What is the mean and the variance for the gain of each player.
- (4) 2.2. Assume that the two players are not allowed to have repeated digits in their numbers. What is the mean and the variance for the gain of each player.
- (3) 2.3. In a slightly different scenario, the two players select a three digit number and if the numbers match in the first place, A pays X dollars to B , and if they match in the third place, B pays X dollars to A (if the two numbers match in both places, neither of them pays). What is the mean and the variance for the gain of each player in parts (2.1) and (2.2) above for this case.

Problem 3: A Professor sets an examination with one question for two students. Of the two students, she knows that Student 1 has a probability of .3 of answering the question correctly and Student 2 has a probability of .5 of answering the question correctly.

- (4) 3.1. She chooses a paper at random and finds it contains a correct answer. What is the probability it is the paper of Student 1?
- (3) 3.2. What is the probability that exactly one of the students got the answer correct?
- (3) 3.3. Her TA actually marked the papers and tells her that exactly one student got the answer correct. What is the probability that Student 1 answered the question correctly?

Problem 4: A semi-conductor factory produces chips each containing 8 logical units. Due to the impurity in the silicon surface, a logical unit may fail to function properly after production. The pdf for the percentage of impurity in the area occupied by a given logical unit is as shown in Fig. 1. Each logical unit fails if the percentage of impurity in its corresponding area is more than 25%. For the sake of simplicity, we assume that the logical units fail independently of each other.

To increase the production efficiency, the chips are designed to have $K \geq 8$ logical units. In the quality control, those with at least 8 units functioning properly are selected and 8 of their good units are given external connection to the pins.

- (3) 4.1. Compute the probability that a given logical unit fails.
- (3) 4.2. Give an expression for the probability that a given chip contains at least 8 good units.
- (4) 4.3. Assume that the probability of making a mistake in measuring the percentage of impurity in silicon surface (and deciding whether a given logical unit is good or not) is equal to 0.1. What is the probability that a given logical unit passes the quality control, and what is the conditional probability that such unit is defective.

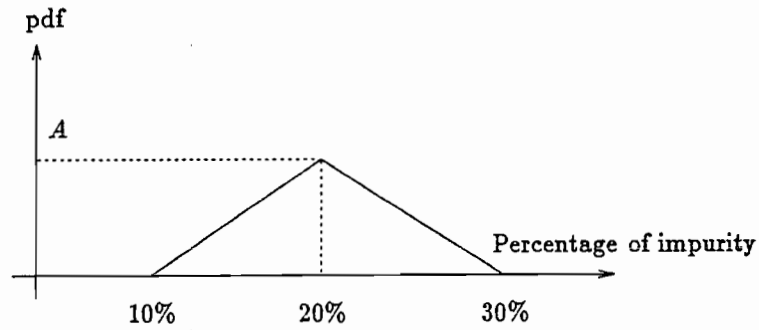


Figure 1: pdf for the percentage of impurity in silicon surface.

University of Waterloo
Department of Electrical and Computer Engineering
E&CE-316 – Introduction to Probability Theory
Midterm Examination
13 June, 1995

Instructor: A. K. Khandani

Time allowed: 1.5 hours.

Closed book examination. One $8\frac{1}{2} \times 11$ review sheet allowed.

Answer any three questions. If all four questions attempted, best three count.

Questions of equal value. [30] constitutes full marks.

Problem 1: In a game of lottery, a player chooses 8 of the numbers from 1 to 40 randomly. The lottery commission then performs an experiment that selects 8 of these 40 numbers randomly. Assuming that the choice of the player and also the choice of the lottery commission is equally likely to be any of the C_{40}^8 combinations, what is the probability that a player has:

- (3) 1.1. Seven of the selected numbers.
- (2) 1.2. All of the 8 numbers selected.
- (2) 1.3. At least 6 of the numbers selected.
- (3) 1.4. What is the probability that 4 of the numbers chosen by the player are from 1 to 9 and 4 of them are from 10 to 15.

Problem 2: Consider the experiment of throwing two dice. A success occurs if both dice show the same number.

- (2) 2.1. What is the probability of success.
- (1) 2.2. What is the probability of obtaining no success in ten attempts.
- (1) 2.3. What is the probability of obtaining at least one success in ten attempts.
- (2) 2.4. Give an expression for the probability of having more than 5 success in 10 attempts.
- (2) 2.5. What is the probability that the first success happens in the n 'th attempt, and compute the mean value of n .
- (2) 2.6. What is the probability that the k 'th success happens in the n 'th attempt, and compute the mean value of n .

Problem 3: Die A has four red and two white faces, whereas die B has two red and four white faces. A fair coin is flipped once and if the result is head, die A is selected, and if it is tail, die B is selected. Then, we throw the selected die for two times in succession.

- (3) 3.1. What is the probability of obtaining two reds.
- (4) 3.2. If we get two reds, what is the probability that die A was selected.
- (3) 3.3. Show that the probability of getting a red in a given throw is $1/2$.

Problem 4: Consider three events A, B, C with $P(A) = 0.6$, $P(B) = 0.5$ and $P(C) = 0.1$. We know that A and B are independent, and C has no outcomes in common with either A or B . Compute the following probabilities:

- (2) 4.1. $P(A \cap B \cap C)$
- (2) 4.2. $P(A \cap B)$, $P(A \cup B)$
- (2) 4.3. $P(A \cap C)$, $P(A \cup C)$
- (2) 4.4. $P(A|B)$, $P(C|B)$.
- (2) 4.5. $P[A|(B \cup C)]$.

University of Waterloo
Department of Electrical and Computer Engineering
Friday, February 16, 1996
4:30pm to 6:00pm.

E&CE-316 Midterm Examination

Instructor: W. Zhuang, 3A Electrical & Computer Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED except hand calculators (A formula sheet is attached to this examination as page 3).

Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

The marking scheme is shown in the left margin and [50] constitutes full marks.
You can earn 5 bonus marks!

- [10] 1. The time interval X between the purchase of a certain video cassette recorder (VCR) and its first trip to the repair shop has approximately the exponential distribution with probability density function (pdf)

$$f(x) = e^{-x}, \quad x > 0$$

where x is in years.

- (a) Find the cumulative distribution function (cdf) for X .
- (b) Estimate the probability that the VCR will work for at least 2 years without requiring any service.
- (c) What is the average time interval between the purchase of the VCR and its first trip to the repair shop?
- (d) What is the variance of X ?

- [5] 2. (a) A boat's crew consists of 8 people, of whom two can row only on the stroke side of the boat and three only on the bow side. In how many ways can the crew be arranged?
- [5] (b) Urn I contains four red chips, three white chips, and two blue chips. Urn II has three red chips, four white chips, and five blue chips. Two chips are drawn at random and without replacement from each urn. What is the probability that all four chips are the same color?
- [5] (c) If ten fair dice are tossed, what is the probability that the sum of their faces is 12?
- [5] (d) n points are chosen at random over the interval $[0, 1]$. What is the minimum value of n in order that there is a probability of 0.95 that at least one of the points is greater than 0.9?
- [10] 3. Two sections of a senior probability course are being taught. From what she has heard about the two instructors listed, Francesca estimates that her chances of passing the course are 0.85 if she gets professor X and 0.60 if she gets professor Y. The section into which she is put is determined by the registrar. Suppose that her chances of being assigned to professor X are 4 out of 10. Fifteen weeks later, we learn that Francesca did, indeed, pass the course. What is the probability that she was enrolled in professor X's section?
- [10] 4. A sandwich vendor parks his truck near the University Library each night to sell snacks, soft drinks, and coffee. He estimates that 20% of the students returning to their dormitories buy snacks, 30% soft drinks, and 10% cups of coffee. He has also observed that if a student buys a snack, there is a 50% chance he or she will also buy a soft drink and a 40% chance he or she will buy some coffee. No one, though, ever buys a soft drink and coffee. Approximately how many of the 200 students walking by the truck on a typical night will buy something?
- [5] 5. **[Bonus]** A line of $2n$ surly Pittsburgh Steeler fans are waiting in front of a ticket window trying to buy \$10 Super Bowl tickets. Half of them have \$10 bills; the other half have \$20 bills. The cashier has no change to start with. What is the probability that those with \$10 bills will be positioned in the queue in such a way that the cashier always has enough change, and all $2n$ fans are able to buy their tickets without being hassled?

Useful facts:

I. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

II. Number of ways to permute k objects from n distinct objects with replacement is

$$n^k.$$

III. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

IV. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

V. Bayes Theorem: Let $\{A_i\}_{i=1}^n$ be a set of events defined over S such that $\bigcup_{i=1}^n A_i = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, and $P(A_i) > 0$ for $i = 1, 2, \dots, n$. For any event B (also defined on S), where $P(B) > 0$,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}.$$

VI. Binomial Distribution: Let p be the probability that event A occurs for a trial. The probability that event A occurs k times in n independent trials ($0 \leq k \leq n$) is

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

VII. "Union" identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

VIII.

$$\begin{aligned} \int x e^{ax} dx &= \frac{e^{ax}}{a^2} (ax - 1) \\ \int x^2 e^{ax} dx &= \frac{e^{ax}}{a^3} (a^2 x^2 - 2ax + 2) \end{aligned}$$

University of Waterloo
Department of Electrical and Computer Engineering
Monday, June 17, 1996
3:30pm to 5:00pm.

E&CE-316 Midterm Examination

Instructor: S. Shen, 3A Electrical Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED except hand calculators (A formula sheet is attached to this examination as page 3).

Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

The marking scheme is shown in the left margin and [50] constitutes full marks.

- [5] 1. (a) A number X is selected at random in the interval $[-1, +2]$. Let the events $A = \{X < 0\}$, $B = \{|X - 0.5| < 1\}$, and $C = \{X > 0.25\}$. Find the probability of B , $A \cap B$, and $A \cup C$.
- [5] (b) Two distinct fair dice are rolled and the numbers on their faces recorded. Find the probability that the sum on the two dice is 7 or 11.
- [20] 2. Light bulbs manufactured according to a certain process are known to last 200 hours on the average. It is assumed that the life time T of a particular bulb has an exponential distribution.
- (a) Find the cumulative distribution function (cdf) of the life time T .
- (b) What fraction of the bulbs manufactured will last longer than 250 hours?
- (c) Bulbs are tested in turn until one is found that lasts longer than 250 hours. What is the probability that this happens for the first time with the seventh bulb tested?
- (d) Each bulb costs \$1.25 to manufacture and is sold for \$2.50. If a bulb burns out before 50 hours, the customer is entitled to a full refund. What is the expected profit per bulb?

- [5] 3. (a) Four men and four women are to be seated along one side of a table. How many arrangements are possible if the men must sit in alternate chairs? How many arrangements are possible if the two end chairs must be occupied by men?
- [5] (b) On one day's production of 800 cars, it is known that 70 are defective. A particular car dealer chooses 40 of these cars at random. Let X denote the number of defective cars that the dealer selects. Find an expression of $E(X)$.
- [5] (c) Each day a stock price moves up one point or down one point with probabilities 0.25 and 0.75, respectively. What is the probability that after 4 days the stock will have returned to its original price? Assume that the daily price fluctuations are independent events.
- [5] 4. Suppose X is a number randomly selected from the interval $(0, 1)$. What is the probability that the second decimal digit in $\sqrt{X}$ is equal to 5?

Formula Sheet:

I. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

II. Number of ways to permute k objects from n distinct objects with replacement is

$$n^k.$$

III. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

IV. Binomial Distribution: Let p be the probability that event A occurs for a trial. The probability that event A occurs k times in n independent trials ($0 \leq k \leq n$) is

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

V. Poisson distribution with parameter $\lambda(> 0)$: $p_X(j) = e^{-\lambda} \lambda^j / j!$, $j = 0, 1, 2, \dots$

VI. Exponential distribution with parameter $\lambda(> 0)$:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

VII. "Union" identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

VIII.

$$\begin{aligned} \int x e^{ax} dx &= \frac{e^{ax}}{a^2} (ax - 1) \\ \int x^2 e^{ax} dx &= \frac{e^{ax}}{a^3} (a^2 x^2 - 2ax + 2) \end{aligned}$$

University of Waterloo
Department of Electrical and Computer Engineering
Thursday, February 20, 1997
5:00pm to 6:30pm.

E&CE-316 Midterm Examination

Instructors: S. Shen, 3A Electrical Engineering
W. Zhuang, 3A Computer Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED except hand calculators (A formula sheet is attached to this examination as page 3).

Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

The marking scheme is shown in the left margin and [50] constitutes full marks.
You can earn 5 bonus marks!

- [15] 1. A continuous random variable X has the probability density function of the form

$$f_X(x) = \begin{cases} A(1 - |x|), & -1 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) For what value of A is this a density function?
- (b) Find the mean, variance, and cdf of X .
- (c) What is the probability that the roots, y_1 and y_2 , of the equation

$$y^2 + y + X = 0$$

will be complex?

- [5] 2. A repeater on a telephone line has a lightning protection circuit. When a lightning flash comes, the repeater has a probability p of surviving it. In a given period of time $(0, t)$, the number of lightning flashes follows a Poisson distribution with parameter λt . What is the probability that the repeater is still functioning after t seconds?

- [5] 3. (a) How many tosses of a fair coin are needed so that the probability of getting at least one head is at least 0.99?
- [5] (b) Five married couples are invited to a banquet and are seated at random around a circular table. What is the probability that each husband will be sitting next to his wife?
- [5] (c) A quality control inspector draws one part at a time without replacement from a set containing 5 defective parts and 10 good parts. What is the probability that the third defective part is found on the eighth drawing?
- [5] (d) Each of two persons tosses 3 fair coins. What is the probability that they obtain the same number of heads?
- [10] 4. Two baseball teams are drawn from grade 6 and 7 students. Initially team 1 has four grade 6 and six grade 7 while team 2 has four grade 6 and four grade 7 players. A player is chosen at random from team 1 and transferred to team 2. If a player is then chosen at random from team 2 to toss a coin, and he happens to be in grade 7, what is the probability that the transferred player was in grade 7?
- [5] 5. [Bonus] Consider the game of heads-tails, in which a coin is thrown and a player wins \$1 if he correctly calls the side of the coin which lands upwards, but otherwise loses \$1. Suppose the player's initial fortune is x (≥ 0) dollars, the intention is to play until m ($\geq x$) dollars have been won, after which the game will terminate. Moreover, the game will also be terminated if the player loses the entire initial fortune before winning m dollars (i.e. the player is "ruined"). What is the probability that the player is ruined?

Useful facts:

1. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

2. Number of ways to permute k objects from n distinct objects with replacement is

$$n^k.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Binomial distribution:

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

5. Poisson distribution with parameter $\lambda(>0)$: $p_X(j) = e^{-\lambda} \lambda^j / j!$, $j = 0, 1, 2, \dots, \infty$.

6. Exponential distribution with parameter $\lambda(>0)$: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

7. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

8. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

9. "Union" identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

10. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where A_i 's form a partition of the sample space.

11. Bayes theorem: $P(A_j|B) = P(B|A_j)P(A_j)/[\sum_{i=1}^n P(B|A_i)P(A_i)]$, where A_i 's form a partition of the sample space.

University of Waterloo
Department of Electrical and Computer Engineering
Friday, June 20, 1997
4:30pm to 6:00pm.

E&CE-316 Midterm Examination

Instructor: S. Shen, 3A Electrical Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED except hand calculators (A formula sheet is attached to this examination as page 3).

Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

The marking scheme is shown in the left margin and [45] constitutes full marks.

[15] 1. Determine whether each of the following statements is true or false and justify your answer.

- (a) The range space R_X is the subset of the real line, therefore $0 < P(R_X) < 1$.
- (b) If A and B are disjoint events in a sample space S , each with nonzero probability, then $P(A|B) = P(A)$.
- (c) Let X be a continuous random variable with a pdf that is symmetric about the origin, then $F_X(x) + F_X(-x) = 1$. ($F_X(x)$ is the cdf of X at real number x).
- (d) If $P(A|B) = 0.4$, $P(A) = 0.2$ and $P(A \cup B) = 0.5$, then $P(B)$ must be 0.6.
- (e) If X is a random variable with probability density function

$$f_X(x) = 3e^{-3(x-a)}, \quad x \geq 2,$$

then $a = 2$.

[5] 2. (a) Calls come into a telephone exchange at an average rate of 1.5 per minute. Assuming that the number of calls received follows a Poisson distribution, find the probability that at least 3 calls are received in the next 4 minutes.

- [5] (b) Let X be a random variable with mean μ and variance σ^2 . For what value of a constant C is $E((X - C)^2)$ minimized?
- [5] 3. (a) A machine produces items with 10% defective. Every hour, on the hour, an inspector chooses ten items from the output of that machine and if he finds two or more defectives he shuts the machine down. What is the expected length of time (in hours) the machine runs between shutdowns?
- [5] (b) In Lotto 649, a player chooses 6 distinct numbers from 1 to 49. This set is compared to 6 other numbers chosen at random by the house and prizes are awarded if 3 or more numbers are matched. What is the probability that the player gets a prize?
- [5] 4. According to the *New York Time* (September 5, 1987), a test for the presence of HIV exists that gives a positive result (indicating the virus) with certainty if a patient actually has the virus. However, associated with this test, there is a *false positive rate*; that is, the test will sometimes indicate the presence of the virus in patients actually free of the virus. This test has a false positive rate of 1 in 20,000. Assuming now that 1 person in 10,000 is actually HIV positive, what proportion of patients for whom the test indicates HIV actually have the virus?
- [5] 5. You have two fair coins. You toss both of them once, and set aside any that come up heads. You continue to toss the coins remaining, on each toss removing those that come up heads, until all of the coins have come up heads. On average, how many (group) tosses will you have to make?

Useful facts:

1. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

2. Number of ways to permute k objects from n distinct objects with replacement is

$$n^k.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Binomial distribution:

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots, n; \quad E(X) = np.$$

5. Poisson distribution with parameter $\lambda (> 0)$: $p_X(j) = e^{-\lambda} \lambda^j / j!$, $j = 0, 1, 2, \dots, \infty$; $E(X) = \lambda$.

6. Exponential distribution with parameter $\lambda (> 0)$: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $E(X) = 1/\lambda$.

7. Geometric distribution with parameter θ : $p_X(i) = (1-\theta)^{i-1} \theta$, $i = 1, 2, \dots, \infty$; $E(X) = 1/\theta$.

8. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

9. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

10. "Union" identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

11. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where A_i 's form a partition of the sample space.

12. Bayes theorem: $P(A_j|B) = P(B|A_j)P(A_j) / [\sum_{i=1}^n P(B|A_i)P(A_i)]$, where A_i 's form a partition of the sample space.

University of Waterloo
Department of Electrical and Computer Engineering
Thursday, February 19, 1998
4:30pm to 6:00pm, DC1350, 1351
E&CE-316 Midterm Examination

Instructor: X.S. Shen, 3A Electrical and Computer Engineering

Time allowed: 1.5 hours.

NO AIDS ALLOWED except hand calculators (A formula sheet is attached to this examination as page 3).

Attempt all the questions. **JUSTIFY ALL YOUR ANSWERS.**

The marking scheme is shown in the left margin and [50] constitutes full marks.

- [10] 1. Determine whether each of the following statements is true or false and justify your answer.
- (a) If $P(A) = 0$, then event A is a null set.
 - (b) If two events A_1 and A_2 of a sample space S are disjoint events, they must form a partition of the sample space.
 - (c) If a continuous random variable X has a uniform distribution (i.e., the pdf of X is a positive constant), then the mean of X is also positive.
 - (d) If events A and B are subsets of a sample space S , then $P(\bar{A} \cap \bar{B}) = 1 - P(A \cap B)$.

- [5] 2. (a) A random variable X is said to have a *Laplace* distribution and its pdf is given by

$$f_X(t) = \frac{1}{2} \lambda e^{-\lambda|t|}, \quad -\infty < t < \infty.$$

Find the cumulative distribution function (cdf) $F_X(t)$.

- [10] (b) The number of particles, X , observed during the time interval $(0, t)$ in a certain experiment has a Poisson distribution with parameter λt , i.e.,

$$P(X = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots, \infty.$$

If $t = 1$ and it is known that the ratio $P(X \leq 2)/P(X = 0)$ is 2.5, what is the average number of particles observed in the interval $(0, 1)$? What is the most likely number of particles to be observed in this interval?

- [10] 3. (a) An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, its policy is to sell 52 tickets for a flight that can only hold 50 passengers. What is the probability that there will be a seat available for every passenger that shows up?

- [5] (b) A somewhat drunk conventioneer finds himself in the embarrassing position of being unable to discern whether he is walking forward or backward — or, what is worse, to predict in which of those directions his next step will be. If he is equally likely to walk forward or backward, what is the probability that after hazarding n such maneuvers, he will have moved forward a distance of r steps?

- [10] 4. A coin is randomly selected from a group of ten coins, the n th coin having a probability $n/10$ of coming heads. The coin is then repeatedly flipped until a head appears. Let N denote the number of flips necessary. What is the probability distribution of N ?

Useful facts:

1. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

2. Number of ways to permute k objects from n distinct objects with replacement is

$$n^k.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Binomial distribution:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}. \quad E(X) = np.$$

5. Poisson distribution: $P(X = j) = e^{-\lambda} \lambda^j / j!$, $j = 0, 1, 2, \dots, \infty$; $\lambda > 0$. $E(X) = \lambda$.

6. Geometric distribution: $P(X = j) = p(1-p)^{j-1}$, $j = 1, 2, \dots, \infty$. $E(X) = 1/p$.

7. Exponential distribution: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. $E(X) = 1/\lambda$.

8. Set operation: $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

9. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

10. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

11. "Union" identity: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

12. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where A_i 's form a partition of the sample space.

13. Bayes theorem: $P(A_j|B) = P(B|A_j)P(A_j)/[\sum_{i=1}^n P(B|A_i)P(A_i)]$, where A_i 's form a partition of the sample space.

E&CE 316: Introduction to Probability Theory
Midterm Examination

Thursday, June 18, 1998
5:00pm to 6:30pm, EL 207 & 211

Instructor: Y. C. Yoon

Time allowed: 1.5 hours

Closed book examination. Hand calculators are allowed. A formula sheet is attached on the final page.

Attempt all the questions and justify each of your answers clearly.

The marking scheme is shown on the left margin. [50] constitutes full marks.

- [6] 1. (a) The game of poker dice is played by simultaneously rolling 5 fair dice. Calculate the probability of having: i) no two alike, ii) one pair, iii) two pairs.
- [4] (b) A system of five antennas lined up in a linear order is functional as long as no two consecutive antennas are defective. Two of the antennas are known to be defective and each possible configuration of antennas is equally likely. What is the probability that the system is functional?
- [8] 2. A telegraph communication system transmits two types of signals: a dot or a dash over a channel. Assume that the channel causes an average of $\frac{2}{5}$ of the dots and $\frac{1}{3}$ of the dashes to be changed. The ratio between the number of transmitted dots and that of transmitted dashes is 5:3. What is the probability that the transmitted signal was a dot if the received signal is a dot?
- [7] 3. (a) The lifetime (in terms of years) of a particular machine can be approximated as a continuous random variable X with the probability density function:

$$f_X(x) = \begin{cases} \beta e^{-x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Find the value of the constant β .
- Find the cumulative distribution function for X .
- Find the probability that the machine lasts up to 3 years.
- Find the k th moment of X .

Useful Formula

1. Number of ways to permute k objects from n distinct objects with replacement is n^k .

2. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Number of ways to choose k objects from n distinct objects with replacement is

$$\binom{n+k-1}{k}.$$

5. "Union" identity for two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

6. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

7. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

8. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.

9. Bayes' theorem: $P(A_j|B) = P(B|A_j)P(A_j)/[\sum_{i=1}^n P(B|A_i)P(A_i)]$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.

10. Binomial distribution: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \dots, n$; $E[X] = np$.

11. Poisson distribution with parameter $\lambda > 0$: $p_X(j) = e^{-\lambda} \frac{\lambda^j}{j!}$, $j = 0, 1, 2, \dots$; $E[X] = \lambda$.

12. Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

where $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\Gamma(n) = (n-1)!$ for each integer $n \geq 1$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

University of Waterloo
Department of Electrical & Computer Engineering

E&CE 316: Introduction to Probability Theory
Midterm Examination

Tuesday, June 15, 1999
4:30pm to 6:30pm, DWE 1501 & 1502

Instructor: Y. C. Yoon

Time allowed: $1\frac{1}{2}$ hours

Closed book examination. Hand calculators are allowed. A formula sheet is attached on the final page.

Attempt all the questions and justify each of your answers clearly.

The marking scheme is shown on the margins. [50] constitutes full marks.

Last Name: _____ First Name: _____

Student ID: _____ Signature: _____

Question #	Marks
1	/5
2	/11
3	/14
4	/15
5	/5
Total	/50

[5]

1. These two problems were studied by Antoine Goussault (le Chevalier de Méré) back in the 17th century.

(a) *The first problem of de Méré:* A gambler throws a fair die four times. What is the probability of at least one six in the four throws. [2]

(b) *The second problem of de Méré:* Consider the roll of two fair dice twenty-four times. What is the probability of at least one roll of two sixes in the twenty-four throws. [3]

[11] 2. Five people, designated as A, B, C, D and E , are seated in a linear order.

(a) Assuming that each possible order is equally likely, what is the probability that

i. there is one person between A and B . [3]

ii. there are three people between A and B . [3]

(b) Repeat part (a) assuming that the five people are seated in a circle. [5]

[14]

3. Pollution of rivers in North America has been a major problem for many years.

Consider the following events for a randomly selected river:

$A = \{\text{The river is polluted.}\}$,

$B = \{\text{A sample of water tested detects pollution.}\}$ and

$C = \{\text{Fishing is permitted.}\}$.

Assume:

$$P(A) = 0.3, \quad P(B|A) = 0.75, \quad P(B|\bar{A}) = 0.2,$$

$$P(C|A \cap B) = 0.2, \quad P(C|\bar{A} \cap B) = 0.15, \quad P(C|A \cap \bar{B}) = 0.8, \quad P(C|\bar{A} \cap \bar{B}) = 0.9$$

(a) Find $P(A \cap B \cap C)$. [3]

(b) Find $P(B)$. [2]

(c) Find $P(\bar{B} \cap C)$. [5]

- (d) Find the probability that the river is polluted given that fishing is permitted [4]
and the sample tested did not detect pollution.

[15]

4. During “busy time” when a telephone exchange is close to capacity, callers encounter difficulty in placing their calls. Consider the number of attempts, X , made until a connection is established during a “busy time.” Let the probability of connection for each attempt be $p = 0.05$.

(a) Find the probability of connection on the fifth attempt.

[2]

(b) Find the cumulative distribution function $F_X(k)$.

[4]

(c) Plot $F_X(\cdot)$ and the probability mass function $P_X(k)$ at the values $k \in \{0, 1, 2, 3\}$. [2]

(d) What is the most likely number of attempts? [2]

(e) Find the mean of X and the mean of $2X + 5$. [2]

- (f) At least how many attempts are needed to ensure a probability of connection greater than 90%? [3]

[5]

5. An investor has \$10,000 to invest among four types of mutual funds. Each investment must be in units of \$1,000 dollars. As an example, all investments could be placed in one type of mutual fund alone.

(a) If all \$10,000 is to be invested, how many different investment strategies are possible? [3]

(b) If each unit of \$1,000 belongs to a different client (ie. if each unit is now distinguishable), how many arrangements are possible? [2]

Useful Formula

1. Number of ways to permute k objects from n distinct objects with replacement is n^k .
2. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Number of ways to choose k objects from n distinct objects with replacement is

$$\binom{n+k-1}{k}.$$

5. "Union" identity for two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

6. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

7. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

8. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.

9. Bayes' theorem: $P(A_j|B) = P(B|A_j)P(A_j) / \left[\sum_{i=1}^n P(B|A_i)P(A_i) \right]$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.

10. Binomial distribution: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \dots, n$; $E[X] = np$.

11. Poisson distribution with parameter $\alpha > 0$: $p_X(j) = e^{-\alpha} \frac{\alpha^j}{j!}$, $j = 0, 1, 2, \dots$; $E[X] = \alpha$.

12. Geometric distribution: $p_X(j) = (1-p)^{j-1}p$, $j = 1, 2, \dots$; $E[X] = 1/p$.

13. $\int_0^\infty x^n e^{-x} dx = n!$

14. $\int x e^{ax} dx = e^{ax}(ax - 1)/a^2$

15. $\sum_{i=1}^m i = \frac{m(m+1)}{2}$, $\sum_{i=0}^m x^i = \frac{1-x^{m+1}}{1-x}$, $x \neq 1$

E&CE 316: Introduction to Probability Theory
Midterm Examination

Monday, February 14, 2000
4:30pm to 6:30pm, EL 103 & EL 105 & EL 110

Instructor: Y. C. Yoon

Time allowed: $1\frac{1}{2}$ hours

Closed book examination. Hand calculators are allowed. A formula sheet is attached on the final page.

Attempt all the questions and justify each of your answers clearly.

The marking scheme is shown on the margins. [45] constitutes full marks.

Last Name: _____

First Name: _____

Student ID: _____

Signature: _____

Question #	Marks
1	/7
2	/6
3	/10
4	/11
5	/11
Bonus question	/1
Total	/45

[7]

1. Prove the following.

(a) $P(A \cap B) \geq P(A) + P(B) - 1.$

[2]

(b) $P(C \cap D^c) = P(C) - P(C \cap D).$

[2]

(c) If $P(E|F) = P(E|F^c)$, then E and F are independent.

[3]

[6]

2. The number of handoffs occurring during a cellular phone connection can be grouped into the three categories of: zero, one or more than one handoff. The probability of a caller being on foot is $p_F = 0.4$. Let the probability of zero, one or more than one handoff be, respectively, p_0 , $p_1 = p_0$ or $p_2 = 0.1$ given that the caller is on foot. Let the probability of zero, one or more than one handoff be, respectively, $q_0 = 0.2$, q_1 or $q_2 = q_1$ given that the caller is not on foot.

(a) Find the probability that no handoff occurs.

[3]

- (b) Find the probability that a caller is on foot given that more than one handoff occurred.

[3]

[10]

3. A random variable X has the probability mass function

$$p_X(j) = \begin{cases} \beta j, & j \in \{1, 2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find β .

[2]

(b) Plot the cumulative distribution function of X .

[2]

(c) Find $E[X]$.

[2]

(d) Find $\text{Var}(3X + 2)$.

[4]

[11]

4. *The matching problem:*

The forever absent-minded Homer writes N letters (each personally addressed to different individuals) and seals them in envelopes before writing the addresses on the envelopes. He then proceeds to write N addresses on the envelopes at random.

- (a) What is the probability that at least one letter is addressed correctly if $N = 2$ [2]

- (b) What is the probability that at least one letter is addressed correctly if $N = 3$ [3]

- (c) What is the probability that at least one letter is addressed correctly for general N . [4]

- (d) What is the probability that at least one letter is addressed correctly as $N \rightarrow \infty$. [2]

[11]

5. A computer network consists of N stations scattered across a number of islands in the South Pacific. Assume that radio message transmissions occur independently of other stations and that they can only occur over synchronized time slots. The probability that a station transmits a message in each time slot is p . If only one station transmits over a time slot, then its message reaches its destination successfully. However, if two or more stations transmit in the same time slot, a “collision” is said to have occurred and the messages are discarded. A message is re-transmitted until it reaches its destination successfully.

(a) What is the probability of a collision?

[3]

(b) What is the most likely number of stations which transmit simultaneously in one time slot? [3]

- (c) Repeat (b) assuming that $N = 10$ and $p = .2$. Assume these values only for [1]
this particular part of the question and part (e).

(d) On average, how many times will a station need to transmit a message? [3]

(e) Repeat (d) assuming that $N = 10$ and $p = .2$. [1]

Bonus Question:

[1]

Bob has a wallet containing either a \$5 bill or a \$100 bill where the presence of either is equally likely. He doesn't know which bill it contains though. His friend Sophia inserts a \$5 bill into his wallet. Bob later removes a bill and notes that it is a \$5 bill. What is the probability that the remaining bill in his wallet is a \$5 bill?

Useful Formula

1. Number of ways to permute k objects from n distinct objects with replacement is n^k .
2. Number of ways to permute k objects from n distinct objects without replacement is

$$P_k^n = \frac{n!}{(n-k)!}.$$

3. Number of ways to choose k objects from n distinct objects without replacement is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

4. Number of ways to choose k objects from n distinct objects with replacement is

$$\binom{n+k-1}{k}.$$

5. "Union" identity for two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
6. "Union" identity for three events:
 $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.
7. For $P(B) \neq 0$, the conditional probability of event A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

8. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.
9. Theorem on total probability: $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.
10. Bayes' theorem: $P(A_j|B) = P(B|A_j)P(A_j) / \left[\sum_{i=1}^n P(B|A_i)P(A_i) \right]$, where the events A_i , $i = 1, 2, \dots, n$ form a partition of the sample space.
11. Binomial distribution: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \dots, n$; $E[X] = np$.

12. Poisson distribution with parameter $\alpha > 0$: $p_X(j) = e^{-\alpha} \frac{\alpha^j}{j!}$, $j = 0, 1, 2, \dots$; $E[X] = \alpha$.

13. Geometric distribution: $p_X(j) = (1-p)^{j-1}p$, $j = 1, 2, \dots$; $E[X] = 1/p$.

$$14. \sum_{i=1}^m i = \frac{m(m+1)}{2}, \quad \sum_{i=0}^m x^i = \frac{1-x^{m+1}}{1-x}, \quad x \neq 1, \quad e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$