E&CE 316 Midterm Examination

Winter 2017, Amir K. Khandani

Time: 90mins.

Attempt all questions.

Allowed aids: one page of formula sheet and use of calculators.

1. (10 Points) A company buys tires from two suppliers, 1 and 2. Supplier 1 has a record of delivering tires containing 10% defectives, whereas supplier 2 has a defective rate of only 5%. Suppose 40% of the current supply came from supplier 1. If a tire is taken from this supply and turns out to be defective, find the probability that it came from Supplier 1.

2. (10 points) Jack goes for shopping with 5 ten-dollar bills and 10 one-dollar bill in his wallet. He spends two of his 15 bills, and then chooses one more from the remaining 13 bills. What is the probability that chosen bill is ten-dollar?

3. (10 points) In a standard deck of 52 cards there are 13 spades, 13 clubs, 13 hearts and 13 diamonds. Five cards are drawn at random. Determine the probability that there are two spades, one heart, one club and one diamond.

4. (10 points) The PMF of a random variable "X" is given by $P(X = k) = \frac{3^{-(k+a)}}{k!}, k = 1,2,3.$
   a) (4 points) For what value of "a" is this a true PMF?
   b) (2 points) Find the variance of "X".
   c) (4 points) Compute the Cumulative Distribution Function (CDF) of 3X-1 .

5. (10 points) A very large box of items is known to contain a fraction θ defective. Let X denote the random variable for the number of items to be inspected to obtain the third defective item.
   a) (5 points) Find the probability distribution of X.
   b) (5 points) Find the mean and the variance of X (hint: use independence property as explained in the class).

①

$P(D|S_1) = 0.1$ (S_1: Supplier 1, D: Defective) $\quad P(D|S_2) = 0.05 \quad P(S_1) = 0.4 \quad P(S_2) = 0.6$

$$P(S_1|D) = ?$$

Bayes' formula:

$$P(S_1|D) = \frac{P(D|S_1)P(S_1)}{P(D)} = \frac{P(D|S_1)P(S_1)}{P(D|S_1)P(S_1)+P(D|S_2)P(S_2)}$$

$$= \frac{0.1\times 0.4}{0.1\times 0.4 + 0.05\times 0.6} = \frac{0.04}{0.07} = \boxed{\frac{4}{7}}$$

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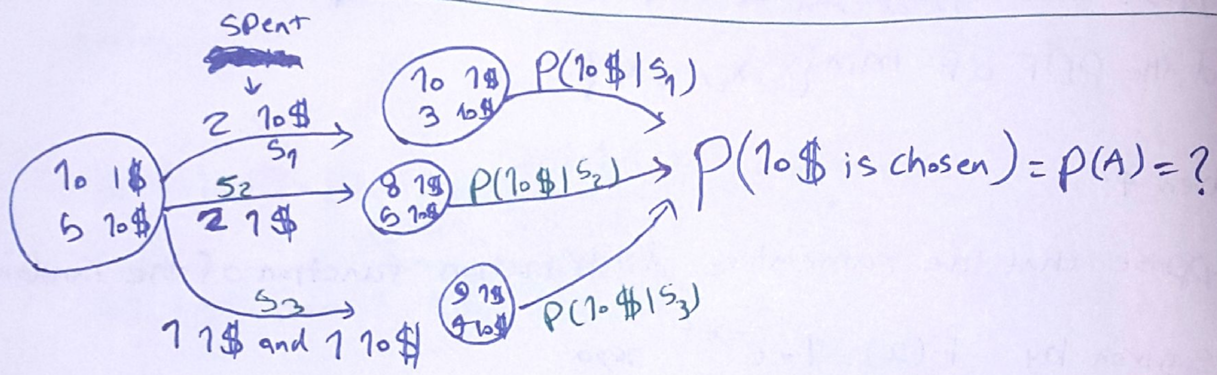
②



You can consider order here or do not consider that.

Two ways:

① Do not consider ordering in spending phase:

$$P(S_1) = \frac{\binom{10}{0}\binom{5}{2}}{\binom{15}{2}} \qquad P(S_2) = \frac{\binom{10}{2}\binom{5}{0}}{\binom{15}{2}} \qquad P(S_3) = \frac{\binom{10}{1}\binom{5}{1}}{\binom{15}{2}}$$

$$P(A|S_1) = \frac{\binom{3}{1}}{\binom{13}{1}} = \frac{3}{13} \qquad P(A|S_2) = \frac{\binom{5}{1}}{\binom{13}{1}} = \frac{5}{13} \qquad P(A|S_3) = \frac{\binom{4}{1}}{\binom{13}{1}} = \frac{4}{13}$$

$$P(A) = P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + P(A|S_3)P(S_3) = \boxed{\frac{1}{3}}$$

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② Consider ordering in spending phase:

$$P(S_1) = \frac{5\times 4}{15\times 14} \qquad P(S_2) = \frac{10\times 9}{15\times 14}$$

$$\left.\begin{array}{l} P(S_{31}) = \dfrac{10\times 5}{15\times 14} \\ \text{or} \\ P(S_{32}) = \dfrac{5\times 10}{15\times 14} \end{array}\right\} \Rightarrow P(S_3) = \frac{2\times 10\times 5}{15\times 14}$$

$$P(A) = \frac{20}{15\times 14}\times\frac{3}{13} + \frac{90}{15\times 14}\times\frac{5}{13} + \frac{100}{15\times 14}\times\frac{4}{13} = \frac{910}{2730} = \boxed{\frac{1}{3}}$$

③

Two ways:

a) Do not consider ordering:

$$\frac{\underbrace{\binom{13}{2}}_{\text{spades}}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{5}} = \frac{2197}{33320} = \boxed{0.0659}$$

a) Consider ordering:

one way:

$$\frac{(13\times 12)\times 13\times 13\times 13\times \dfrac{(2+1+1+1)!}{\underbrace{2!}_{\text{spades}}(1!)^3}}{52\times 51\times 50\times 49\times 48}$$

4 Types

5 items

- 2 Spades
- 1 Clubs
- 1 Heart
- 1 Diamond

$$= \frac{\left(\frac{13\times 12}{2!}\right)\times\binom{13}{1}\times\binom{13}{1}\times\binom{13}{1}}{\dfrac{52\times 51\times 50\times 49\times 48\times 47!}{5!\times 47!}} = \frac{\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{5}} \checkmark = 0.0659$$

other way: (Restriction considering Strategy "First session")

Putting spades and heart together

$$\frac{\overbrace{13\times 12}^{\text{spades}}\times\left(\dfrac{\cancel{3!}}{2!\,1!}\right)\times \overset{\text{Heart}}{13}\times\dfrac{\cancel{4!}}{\underbrace{\cancel{3!}}_{\text{Heart \& spades}}1!}\times 13\times\dfrac{5!}{\cancel{4!}\,1!}\times 13}{52\times 51\times 50\times 49\times 48} = \frac{\frac{13\times 12}{2!}\times\binom{13}{1}^3}{\dfrac{52\times 51\times 50\times 49\times 48}{5!}}$$

$$= \frac{\binom{13}{2}\binom{13}{1}^3}{\binom{52}{5}} \checkmark = 0.0659$$

4. $3^{-a} \times \left(\frac{3^{-1}}{1!} + \frac{3^{-2}}{2!} + \frac{3^{-3}}{3!} \right) =$

$$3^{-a} \times \left(\frac{1}{3} + \frac{1}{18} + \frac{1}{162} \right) = \frac{3^{-a}(54+9+1)}{162} = \frac{64}{162} 3^{-a} = 1$$

$$3^{-a} = \frac{162}{64} \Rightarrow a = \log_3 \frac{64}{162} = -0.845$$

b) $E(X) = (1)P(X=1) + (2)P(X=2) + (3)P(X=3) =$

$$(1)\,3^{-(1-0.845)} + \frac{(2)\,3^{-(2-0.845)}}{2!} + \frac{(3)\,3^{-(3-0.845)}}{3!} =$$

$$0.843 + 0.281 + 0.046 = 1.17$$

$$E(X^2) = 0.843 + 2 \times (0.281) + 3 \times 0.046 = 1.543$$

$$Var(X) = E(X^2) - (E(X))^2 = 1.543 - 1.368 = 0.175$$

$$CDF(X) = \begin{cases} 0 & X < 1 \\ P(X=1) & 1 \leq X < 2 \\ P(X=1) + P(X=2) & 2 \leq X < 3 \\ P(X=1) + P(X=2) + P(X=3) = 1 & 3 \leq X \end{cases}$$

$Y = 3X - 1$

$$F(y) = P(Y \leq y) = P(3X - 1 \leq y) = P\left(X \leq \frac{y+1}{3}\right)$$

$$CDF(Y) = \begin{cases} 0 & \frac{y+1}{3} < 1 \longrightarrow y < 2 \\ 0.843 & 1 \leq \frac{y+1}{3} < 2 \longrightarrow 2 \leq y < 5 \\ 0.984 & 2 \leq \frac{y+1}{3} < 3 \longrightarrow 5 \leq y < 8 \\ 1 & 3 \leq \frac{y+1}{3} \qquad 8 \leq y \end{cases}$$

5) a) we try to find $P\{X=n\}$ where X is random variable for the number of items to be inspected to find 3^{rd} defective items.

among first n-1 items, 2 items should be defective

and the n^{th} item is defective too.

$$\Rightarrow P\{X=n\} = \underbrace{\left(\binom{n-1}{2}\theta^2(1-\theta)^{n-3}\right)}_{\text{first } n-1 \text{ ones}} \cdot \underbrace{\theta}_{\substack{\text{last}\\ \text{one}}} = \binom{n-1}{2}\theta^3(1-\theta)^{n-3}$$

b) define r.v Y_1 such that it describe number to be inspected to find first defective items. $\longrightarrow$ it has geometric dist.

$$\Rightarrow E(Y_1) = \frac{1}{\theta}, \quad Var(Y_1) = \frac{1-\theta}{\theta^2}$$

after finding first one, for finding second defective items, define Y_2. again it has geometric dist.

and similarily for the 3^{rd} one define Y_3.

$$E(Y_2) = E(Y_3) = \frac{1}{\theta}, \quad var(Y_2) = var(Y_3) = \frac{1-\theta}{\theta^2}$$

because they are independenty

$$\Longrightarrow E(X) = E(Y_1+Y_2+Y_2) = E(Y_1)+E(Y_2)+E(Y_3) = \frac{3}{\theta}$$

$$Var(X) = Var(Y_1+Y_2+Y_3) = var(Y_1)+var(Y_2)+var(Y_3) = \frac{3(1-\theta)}{\theta^2}$$