Problem 1 (counting license plates)

Part (a): In each of the first two places we can put any of the 26 letters giving 26^2 possible letter combinations for the first two characters. Since the five other characters in the license plate must be numbers, we have 10^5 possible five digit letters their specification giving a total of

$$26^2 \cdot 10^5 = 67600000\,,$$

total license plates.

Part (b): If we can't repeat a letter or a number in the specification of a license plate then the number of license plates becomes

$$26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 19656000\,,$$

total license plates.

Problem 11 (counting arrangements of books)

Part (a): We have $(3+2+1)! = 6! = 720$ arrangements.

Part (b): The mathematics books can be arranged in 2! ways and the novels in 3! ways. Then the block ordering of mathematics, novels, and chemistry books can be arranged in 3! ways resulting in

$$(3!) \cdot (2!) \cdot (3!) = 72\,,$$

possible arrangements.

Part (c): The number of ways to arrange the novels is given by $3! = 6$ and the other three books can be arranged in 3! ways with the blocks of novels in any of the four positions in between giving

$$4 \cdot (3!) \cdot (3!) = 144\,,$$

possible arrangements.

11)

3 Novels, 2 Mathematics and 1 chemistry

a.) $\boxed{(3+2+1)! = 6!}$

b) $\underbrace{N_1 N_2 N_3}_{\downarrow}$ $\underbrace{M_1 M_2}_{\downarrow}$ $\underbrace{Ch_1}_{\downarrow}$

N_{bundle} M_b Ch_b $\Rightarrow$ ordering $= 3!$

inside N_{bundle}: 3 items $\Rightarrow$ 3! orderings Mathematics: 2! Chemistry: 1!

Result $= (3!\ 2!\ 1!)(3!)$

c) $\underbrace{N_1 N_2 N_3}$

N_{bundle} $M_1 M_2$ Ch_1 $\Rightarrow$ 4! Inside N_{bundle} $\Rightarrow$ 3!

$\boxed{4! \times 3!}$

Problem 13 (counting handshakes)

With 20 people the number of pairs is given by

$$\binom{20}{2} = 190\,.$$

Problem 17 (distributing gifts)

We can choose seven children to give gifts to in $\binom{10}{7}$ ways. Once we have chosen the seven children, the gifts can be distributed in 7! ways. This gives a total of

$$\binom{10}{7} \cdot 7! = 604800\,,$$

possible gift distributions.

Problem 19 (counting committee's with constraints)

Part (a): We select three men from six in $\binom{6}{3}$, but since two men won't serve together we need to compute the number of these pairings of three men that have the two that won't serve together. The number of committees we can form (with these two together) is given by

$$\binom{2}{2} \cdot \binom{4}{1} = 4\,.$$

So we have

$$\binom{6}{3} - 4 = 16\,,$$

possible groups of three men. Since we can choose $\binom{8}{3} = 56$ different groups of women, we have in total $16 \cdot 56 = 896$ possible committees.

Part (b): If two women refuse to serve together, then we will have $\binom{2}{2} \cdot \binom{6}{1}$ groups with these two women in them from the $\binom{8}{3}$ ways to draw three women from eight. Thus we have

$$\binom{8}{3} - \binom{2}{2} \cdot \binom{6}{1} = 56 - 6 = 50\,,$$

possible groupings of woman. We can select three men from six in $\binom{6}{3} = 20$ ways. In total then we have $50 \cdot 20 = 1000$ committees.

Part (c): We have $\binom{8}{3} \cdot \binom{6}{3}$ total committees, and

$$\binom{1}{1} \cdot \binom{7}{2} \cdot \binom{1}{1} \cdot \binom{5}{2} = 210\,,$$

committees containing the man and women who refuse to serve together. So we have

$$\binom{8}{3} \cdot \binom{6}{3} - \binom{1}{1} \cdot \binom{7}{2} \cdot \binom{1}{1} \cdot \binom{5}{2} = 1120 - 210 = 910\,,$$

total committees.

Problem 20 (counting the number of possible parties)

Part (a): There are a total of $\left(\begin{array}{c}8\\5\end{array}\right)$ possible groups of friends that could attend (assuming no feuds). We have $\left(\begin{array}{c}2\\2\end{array}\right)\cdot\left(\begin{array}{c}6\\3\end{array}\right)$ sets with our two feuding friends in them, giving

$$\left(\begin{array}{c}8\\5\end{array}\right)-\left(\begin{array}{c}2\\2\end{array}\right)\cdot\left(\begin{array}{c}6\\3\end{array}\right)=36$$

possible groups of friends

Part (b): If two fiends must attend together we have that $\left(\begin{array}{c}2\\2\end{array}\right)\left(\begin{array}{c}6\\3\end{array}\right)$ if the *do* attend the party together and $\left(\begin{array}{c}6\\5\end{array}\right)$ if they *don't* attend at all, giving a total of

$$\left(\begin{array}{c}2\\2\end{array}\right)\left(\begin{array}{c}6\\3\end{array}\right)+\left(\begin{array}{c}6\\5\end{array}\right)=26\,.$$

Problem 21 (number of paths on a grid)

From the hint given that we must take four steps to the right and three steps up, we can think of any possible path as an arraignment of the letters "U" for up and "R" for right. For example the string

$$UUURRRR\,,$$

would first step up three times and then right four times. Thus our problem becomes one of counting the number of unique arrangements of three "U"'s and four "R"'s, which is given by

$$\frac{7!}{4!\cdot 3!}=35\,.$$

Problem 22 (paths on a grid through a specific point)

One can think of the problem of going through a specific point (say P) as counting the number of paths from the start A to P and then counting the number of paths from P to the end B. To go from A to P (where P occupies the $(2,2)$ position in our grid) we are looking for the number of possible unique arrangements of two "U"'s and two "R"'s, which is given by

$$\frac{4!}{2! \cdot 2!} = 6 \,,$$

possible paths. The number of paths from the point P to the point B is equivalent to the number of different arrangements of two "R"'s and one "U" which is given by

$$\frac{3!}{2! \cdot 1!} = 3 \,.$$

From the basic principle of counting then we have $6 \cdot 3 = 18$ total paths.

28) we consider schools as labels for teacher S of for part:

a)

one example: $S_1 \quad S_1 \quad S_1 \quad S_1 \quad S_1 \quad S_2 \quad S_3 \quad S_3$

School 1

So for each teacher there are 4 possible labels $\{S_1, S_2, S_3, S_4\}$:

$$\underbrace{4 \times 4 \times \cdots \times 4}_{8 \text{ teacher}} = (4)^8$$

b) In this part only two teachers can have same label one example:

$\boxed{S_1}\boxed{S_1}\boxed{S_2}\ \boxed{S_3}\ \boxed{S_4}\ \boxed{S_2}\ \boxed{S_4}\ \boxed{S_3}$

So we can think about this problem in a different way:

We have 4 types: $\{S_1, S_2, S_3, S_4\}$

$\quad\quad\quad\quad\quad\quad\quad$ 2 items 2 2 2

or suppose: $S_1 = A \quad S_2 = B \quad S_3 = C \quad S_4 = D$ and we want to find # of all possible letters arrangement of: AABBCCDD

Result is $= \dfrac{\overset{\text{\# of labels}}{8!}}{(2!)(2!)(2!)(2!)}$

Problem 28 (divisions of teachers)

If we decide to send n_1 teachers to school one and n_2 teachers to school two, etc. then the total number of unique assignments of (n_1, n_2, n_3, n_4) number of teachers to the four schools is given by

$$\binom{8}{n_1\,,n_2\,,n_3\,,n_4}\,.$$

Since we want the total number of divisions, we must sum this result for all possible combinations of n_i, or

$$\sum_{n_1+n_2+n_3+n_4=8}\binom{8}{n_1\,,n_2\,,n_3\,,n_4} = (1+1+1+1)^8 = 65536\,,$$

possible divisions.

If each school must receive two in each school, then we are looking for

$$\binom{8}{2\,,2\,,2\,,2} = \frac{8!}{(2!)^4} = 2520\,,$$

orderings.

29) we look ranks as spots:

First rank → $\square\ \square\ \square \cdots \square\ \square\ \square$

Only countries are distinguishable, so one example is:

U U U R R Ch, Ca Ch R R

So we have 10 items with 4 types: $\dfrac{10!}{\underset{Us}{(3!)}\,\underset{R}{(4!)}\,\underset{Ch}{(2!)}\,\underset{Ca}{(1!)}}$

b) $\underbrace{\square\ \square\ \square}_{1\text{ U here}} \cdots \underbrace{\square\ \square\ \square}_{2\text{ U's here}}$

$$\binom{3}{1} \times \binom{3}{2}$$

one example: $\square\ \boxed{U}\ \square \cdots \boxed{U}\ \square\ \boxed{U}$

5

Now we have 7 remaining spots: $\square\ \square \cdots \square$

No we have 7 items with 3 types: $\{R, ch, ca\} \Rightarrow \dfrac{7!}{(4)!\,(2!)(1!)}$

Problem 29 (dividing weight lifters)

We have 10! possible permutations of all weight lifters but the permutations of individual countries (contained within this number) are irrelevant. Thus we can have

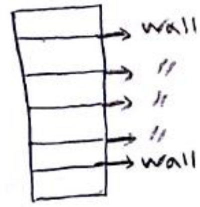
$$\frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!} = \left(\begin{array}{c} 10 \\ 3\,,4\,,2\,,1 \end{array} \right) = 12600\,,$$

possible divisions. If the united states has one competitor in the top three and two in the bottom three. We have $\left(\begin{array}{c} 3 \\ 1 \end{array} \right)$ possible positions for the US member in the first three positions and $\left(\begin{array}{c} 3 \\ 2 \end{array} \right)$ possible positions for the two US members in the bottom three positions, giving a total of

$$\left(\begin{array}{c} 3 \\ 1 \end{array} \right) \left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3 \cdot 3 = 9\,,$$

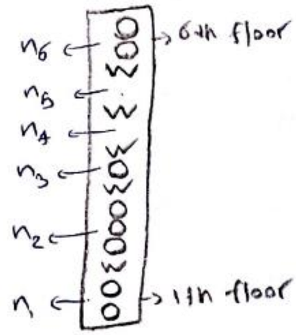
combinations of US members in the positions specified. We also have to place the other countries participants in the remaining $10 - 3 = 7$ positions. This can be done in $\left(\begin{array}{c} 7 \\ 4\,,2\,,1 \end{array} \right) = \frac{7!}{4! \cdot 2! \cdot 1!} = 105$ ways. So in total then we have $9 \cdot 105 = 945$ ways to position the participants.

32) a) we have 8 same people and 6 floors →



{00000000}

one possible outcome:



$$\boxed{n_1+n_2+n_3+n_4+n_5+n_6=8}$$

All possible non-negative Solutions for this formula

(could be zero)

Or all possible letter arrangements of these letters: $\underbrace{00000000}_{8}\ \underbrace{wwwww}_{5}$

$$\frac{13!}{8!\,5!}=\binom{13}{8}=\binom{13}{5}$$

b) For this part we can devide proble into two parallel problems:

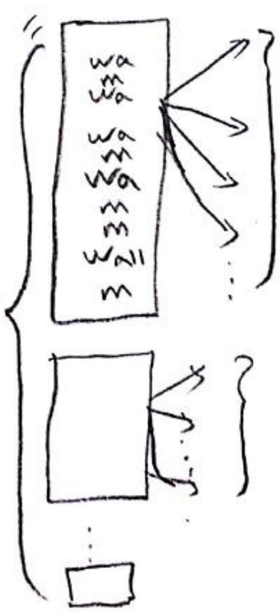
(I) 5 men, No women & 6 floors

(II) 3 women, " men "

(III) (I) ⊗ (II) → why?

Example:

# of possible ways for men to leave



# of possible ways for women to leave

⇒ Result = # of ways men × # of ways women

For men: $\left\{\begin{array}{l} 5 \text{ items with type (man)} \\ 5 \text{ '' '' '' (wall)} \end{array}\right\} \Rightarrow \dfrac{10!}{5!\,5!} = \binom{10}{5}$

$\Rightarrow$ All possible ways $= \binom{10}{5}\binom{8}{5}$

For Women: $\left\{\begin{array}{l} 3 \text{ items with type (woman)} \\ 5 \text{ '' '' '' (walls)} \end{array}\right. \Rightarrow \dfrac{8!}{3!\,5!} = \binom{8}{5}$

** : Why following approach is wrong? **

First we find all possible ways for men to leave $= \binom{10}{5}$

After that we assume that we have new type (Men & walls) and we find all possible ways for women to leave: $\binom{13}{3} = \binom{13}{10} = \dfrac{13!}{\underbrace{10!}_{\text{(Men \& walls)}}\,\underbrace{3!}_{\text{(women)}}}$

Result: $\binom{10}{5}\binom{13}{10}$

Problem 32 (distributing people)

Assuming that the elevator operator can only tell the number of people getting off at each floor, we let x_i equal the number of people getting off at floor i, where $i = 1, 2, 3, 4, 5, 6$. Then the constraint that all people are off at the sixth floor means that $\sum_i x_i = 8$, with $x_i \geq 0$. This has

$$\left(\begin{array}{c} n+r-1 \\ r-1 \end{array} \right) = \left(\begin{array}{c} 8+6-1 \\ 6-1 \end{array} \right) = \left(\begin{array}{c} 13 \\ 5 \end{array} \right) = 1287\,,$$

possible distribution people. If we have five men and three women, let m_i and w_i be the number of men and women that get off at floor i. We can solve this problem as the combination of two problems. That of tracking the men that get off on floor i and that of tracking the women that get off on floor i. Thus we must have

$$\begin{aligned} \sum_{i=1}^{6} m_i &= 5 \quad m_i \geq 0 \\ \sum_{i=1}^{6} w_i &= 3 \quad w_i \geq 0\,. \end{aligned}$$

The number of solutions to the first equation is given by

$$\left(\begin{array}{c} 5+6-1 \\ 6-1 \end{array} \right) = \left(\begin{array}{c} 10 \\ 5 \end{array} \right) = 252\,,$$

while the number of solutions to the second equation is given by

$$\left(\begin{array}{c} 3+6-1 \\ 6-1 \end{array} \right) = \left(\begin{array}{c} 8 \\ 5 \end{array} \right) = 56\,.$$

So in total then (since each number is exclusive) we have $252 \cdot 56 = 14114$ possible elevator situations.

Problem 33 (possible investment strategies)

Let x_i be the number of investments made in opportunity i. Then we must have

$$\sum_{i=1}^{4} x_i = 20$$

with constraints that $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$. Writing this equation as

$$x_1 + x_2 + x_3 + x_4 = 20$$

we can subtract the lower bound of each variable to get

$$(x_1 - 2) + (x_2 - 2) + (x_3 - 3) + (x_4 - 4) = 20 - 2 - 2 - 3 - 4 = 9\,.$$

Then defining $v_1 = x_1 - 2$, $v_2 = x_2 - 2$, $v_3 = x_3 - 3$, and $v_4 = x_4 - 4$, then our equation becomes $v_1 + v_2 + v_3 + v_4 = 9$, with the constraint that $v_i \geq 0$. The number of solutions to equations such as these is given by

$$\left(\begin{array}{c} 9+4-1 \\ 4-1 \end{array} \right) = \left(\begin{array}{c} 12 \\ 3 \end{array} \right) = 220\,.$$

Part (b): First we pick the three investments from the four possible in $\binom{4}{3} = 4$ possible ways. The four choices are denoted in table 1, where a one denotes that we invest in that option. Then investment choice number one requires the equation $v_2+v_3+v_4 = 20-2-3-4 =$

choice	$v_1 = x_1 - 2 \geq 0$	$v_2 = x_2 - 2 \geq 0$	$v_3 = x_3 - 3 \geq 0$	$v_4 = x_4 - 4 \geq 0$
1	0	1	1	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	0

Table 1: All possible choices of three investments.

11, and has $\binom{11+3-1}{3-1} = \binom{13}{2} = 78$ possible solutions. Investment choice number two requires the equation $v_1+v_3+v_4 = 20-2-3-4 = 11$, and again has $\binom{11+3-1}{3-1} = \binom{13}{2} = 78$ possible solutions. Investment choice number three requires the equation $v_1+v_2+v_4 = 20-2-2-4 = 12$, and has $\binom{12+3-1}{3-1} = \binom{14}{2} = 91$ possible solutions. Finally, investment choice number four requires the equation $v_1+v_2+v_3 = 20-2-2-3 = 13$, and has $\binom{13+3-1}{3-1} = \binom{15}{2} = 105$ possible solutions. Of course we could also invest in all four opportunities which has the same number of possibilities as in part (a) or 220. Then in total since we can do any of these choices we have $220 + 105 + 91 + 78 + 78 = 572$ choices.

Problem 1 (counting arrangements of letters)

Part (a): Consider the pair of A with B as one object. Now there are two orderings of this "fused" object i.e. AB and BA. The remaining letters can be placed in $4!$ orderings and once an ordering is specified the fused A/B block can be in any of the five locations around the permutation of the letters $CDEF$. Thus we have $2 \cdot 4! \cdot 5 = 240$ total orderings.

Part (b): We want to enforce that A must be before B. Lets begin to construct a valid sequence of characters by first placing the other letters $CDEF$, which can be done in $4! = 24$ possible ways. Now consider an arbitrary permutation of $CDEF$ such as $DFCE$. Then if we place A in the left most position (such as as in $ADFCE$), we see that there are five possible locations for the letter B. For example we can have $ABDFCE$, $ADBFCE$, $ADFBCE$, $ADFCBE$, or $ADFCEB$. If A is located in the second position from the left (as in $DAFCE$) then there are four possible locations for B. Continuing this logic we see that we have a total of $5+4+3+2+1 = \frac{5(5+1)}{2} = 15$ possible ways to place A and B such that they are ordered with A before B in each permutation. Thus in total we have $15 \cdot 4! = 360$ total orderings.

Part (c): Lets solve this problem by placing A, then placing B and then placing C. Now we can place these characters at any of the six possible character locations. To explicitly specify their locations lets let the integer variables n_0, n_1, n_2, and n_3 denote the number of blanks (from our total of six) that are before the A, between the A and the B, between the B and the C, and after the C. By construction we must have each n_i satisfy

$$n_i \geq 0 \quad \text{for} \quad i = 0, 1, 2, 3\,.$$

In addition the sum of the n_i's plus the three spaces occupied by A, B, and C must add to six or

$$n_0 + n_1 + n_2 + n_3 + 3 = 6\,,$$

or equivalently

$$n_0 + n_1 + n_2 + n_3 = 3\,.$$

The number of solutions to such integer equalities is discussed in the book. Specifically, there are

$$\binom{3+4-1}{4-1} = \binom{6}{3} = 20\,,$$

such solutions. For each of these solutions, we have $3! = 6$ ways to place the three other letters giving a total of $6 \cdot 20 = 120$ arrangements.

Part (d): For this problem A must be before B and C must be before D. Let begin to construct a valid ordering by placing the letters E and F first. This can be done in two ways EF or FE. Next lets place the letters A and B, which if A is located at the left most position as in AEF, then B has three possible choices. As in Part (b) from this problem there are a total of $3 + 2 + 1 = 6$ ways to place A and B such that A comes before B. Following the same logic as in Part (b) above when we place C and D there are $5 + 4 + 3 + 2 + 1 = 15$ possible placements. In total then we have $15 \cdot 6 \cdot 2 = 180$ possible orderings.

Part (e): There are $2!$ ways of arranging A and B, $2!$ ways of arranging C and D, and $2!$ ways of arranging the remaining letters E and F. Lets us first place the blocks of letters consisting of the pair A and B which can be placed in any of the positions around E and F. There are three such positions. Next lets us place the block of letters consisting of C and D which can be placed in any of the four positions (between the E, F individual letters, or the A and B block). This gives a total number of arrangements of

$$2! \cdot 2! \cdot 2! \cdot 3 \cdot 4 = 96\,.$$

Part (f): E can be placed in any of five choices, first, second, third, fourth or fifth. Then the remaining blocks can be placed in $5!$ ways to get in total $5(5!) = 600$ arrangement's.

Problem 9 (selecting three students from three classes)

Part (a): To choose three students from $3n$ total students can be done in $\binom{3n}{3}$ ways.

Part (b): To pick three students from the same class we must first pick the class to draw the student from. This can be done in $\binom{3}{1} = 3$ ways. Once the class has been picked we have to pick the three students in from the n in that class. This can be done in $\binom{n}{3}$ ways. Thus in total we have

$$3\binom{n}{3},$$

possible selections of three students all from one class.

Part (c): To get two students in the same class and another in a different class, we must first pick the class from which to draw the two students from. This can be done in $\binom{3}{1} = 3$ ways. Next we pick the other class from which to draw the singleton student from. Since there are two possible classes to select this student from this can be done in two ways. Once both of these classes are selected we pick the individual two and one students from their respective classes in $\binom{n}{2}$ and $\binom{n}{1}$ ways respectively. Thus in total we have

$$3 \cdot 2 \cdot \binom{n}{2}\binom{n}{1} = 6n\frac{n(n-1)}{2} = 3n^2(n-1),$$

ways.

Part (d): Three students (all from a different class) can be picked in $\binom{n}{1}^3 = n^3$ ways.

Part (e): As an identity we have then that

$$\binom{3n}{3} = 3\binom{n}{3} + 3n^2(n-1) + n^3 .$$

We can check that this expression is correct by expanding each side. Expanding the left hand side we find that

$$\binom{3n}{3} = \frac{3n!}{3!(3n-3)!} = \frac{3n(3n-1)(3n-2)}{6} = \frac{9n^3}{2} - \frac{9n^2}{2} + n .$$

While expanding the right hand side we find that

$$\begin{aligned}
3\binom{n}{3} + 3n^2(n-1) + n^3 &= 3\frac{n!}{3!(n-3)!} + 3n^3 - 3n^2 + n^3 \\
&= \frac{n(n-1)(n-2)}{2} + 4n^3 - 3n^2 \\
&= \frac{n(n^2-3n+2)}{2} + 4n^3 - 3n^2 \\
&= \frac{n^3}{2} - \frac{3n^2}{2} + n + 4n^3 - 3n^2 \\
&= \frac{9n^3}{2} - \frac{9n^2}{2} + n ,
\end{aligned}$$

which is the same, showing the equivalence.

Problem 10 (counting five digit numbers with no triple counts)

Lets first enumerate the number of five digit numbers that can be constructed with no repeated digits. Since we have nine choices for the first digit, eight choices for the second digit, seven choices for the third digit etc. We find the number of five digit numbers with no repeated digits given by $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{9!}{4!} = 15120$.

Now lets count the number of five digit numbers where *one* of the digits $1, 2, 3, \cdots, 9$ repeats. We can pick the digit that will repeat in nine ways and select its position in the five digits in $\left(\begin{array}{c} 5 \\ 2 \end{array}\right)$ ways. To fill the remaining three digit location can be done in $8 \cdot 7 \cdot 6$ ways. This gives in total

$$9 \cdot \left(\begin{array}{c} 5 \\ 2 \end{array}\right) \cdot 8 \cdot 7 \cdot 6 = 30240 \,.$$

Lets now count the number five digit numbers with two repeated digits. To compute this we might argue as follows. We can select the first digit and its location in $9 \cdot \left(\begin{array}{c} 5 \\ 2 \end{array}\right)$ ways. We can select the second repeated digit and its location in $8 \cdot \left(\begin{array}{c} 3 \\ 2 \end{array}\right)$ ways. The final digit can be selected in seven ways, giving in total

$$9 \left(\begin{array}{c} 5 \\ 2 \end{array}\right) \cdot 8 \left(\begin{array}{c} 3 \\ 2 \end{array}\right) \cdot 7 = 15120 \,.$$

We note, however, that this analysis (as it stands) double counts the true number of five digits numbers with two repeated digits. This is because in first selecting the first digit from nine classes and then selecting the second digit from eight choices the total two digits chosen can actually be selected in the opposite order but placed in same spots from among our five digits. Thus we have to divide the above number by two giving

$$\frac{15120}{2} = 7560 \,.$$

So in total we have by summing up all these mutually exclusive events we find that the total number of five digit numbers allowing repeated digits is given by

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 + 9 \left(\begin{array}{c} 5 \\ 2 \end{array}\right) \cdot 8 \cdot 7 \cdot 6 + \frac{1}{2} \cdot 9 \cdot \left(\begin{array}{c} 5 \\ 2 \end{array}\right) 8 \left(\begin{array}{c} 3 \\ 2 \end{array}\right) \cdot 7 = 52920 \,.$$

Problem 17 (a simple combinatorial identity)

To show that

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2} \quad \text{for} \quad 1 \leq k \leq n\,,$$

is true, begin by expanding the right hand side (RHS) of this expression. Using the definition of the binomial coefficients we obtain

$$\begin{aligned} \text{RHS} &= \frac{k!}{2!(k-2)!} + k(n-k) + \frac{(n-k)!}{2!(n-k-2)!} \\ &= \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2} \\ &= \frac{1}{2}\left(k^2 - k + kn - k^2 + n^2 - nk - n - kn + k^2 + k\right) \\ &= \frac{1}{2}\left(n^2 - n\right)\,. \end{aligned}$$

Which we can recognize as equivalent to $\binom{n}{2}$ since from its definition we have that

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}\,.$$

proving the desired equivalence. A combinatorial argument for this expression can be given in the following way. The left hand side $\binom{n}{2}$ represents the number of ways to select two items from n. Now for any k (with $1 \leq k \leq n$) we can think about the entire set of n items as being divided into two parts. The first part will have k items and the second part will have the remaining $n-k$ items. Then by considering all possible halves the two items selected could come from will yield the decomposition shown on the right hand side of the above. For example, we can draw our two items from the initial k in the first half in $\binom{k}{2}$ ways, from the second half (which has $n-k$ elements) in $\binom{n-k}{2}$ ways, or by drawing one element from the set with k elements and another element from the set with $n-k$ elements, in $k(n-k)$ ways. Summing all of these terms together gives

$$\binom{k}{2} + k(n-k) + \binom{n-k}{2} \quad \text{for} \quad 1 \leq k \leq n\,,$$

as an equivalent expression for $\begin{pmatrix} n \\ 2 \end{pmatrix}$.