1) Suppose that we have two decks of 52 cards. First we choose 5 cards from first deck and then put them inside second deck. There are how many possible orderings in total, for second deck of cards?

For solving this problem we can think in this way:

First step: we assume that we know 5 chosen cards and we try to find possible orderings of cards in second deck (57 cards)

Second step: we find all possible ways for choosing 5 cards and multiply that by number of result of first step.

First step: suppose that chosen cards are: 4 Aces and King of Spades

In a deck of cards we have $\underline{\underline{52}}$ different types; After adding these 6 cards 5 of these types will have 2 items and 47 types will have 1 item.

So all possible orderings:

$$\frac{\overbrace{57!}^{\#\text{ of all items}}}{\underbrace{(1!)^{47}}_{\text{47 types have one item}}\ \underbrace{(2!)^{5}}_{\text{5 types have 2 items}}}$$

Second step: All possible ways for choosing 5 cards from first deck of cards:

$$\binom{52}{5}$$

Total # of orderings for second deck after adding 6 cards $= \dfrac{57!}{(2!)^5}\dbinom{52}{5}$

P1

2) Find number of all possible orderings of cards in a deck of 52 cards if:

a) Only suit of a card is distinguishable:

$$\frac{52!}{(13!)^4}$$

$13!$: # of Items; 4: # of Types

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b) Only colour of cards are distinguishable.

$$\frac{52!}{(26!)^2}$$

$26 = 2 \times 13$

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c) Suits of cards are distinguishable and one card is removed from deck:

Removed card has one of the possible suits so:

$$\binom{4}{1} \frac{51!}{(13!)^3 (12!)}$$

$\binom{4}{1}$: Choosing which suit not be complete

$(12!)$: One type has only 12 items and others are complete

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d) Suits are distinguishable and two cards are removed;

Two possible ways: (1.) Two removed cards have same suits (2.) Have different suits

(1.) $\binom{4}{1} \dfrac{50!}{(13!)^3 (11!)}$ (2.) $\binom{4}{2} \dfrac{50!}{(13!)^2 (12!)^2}$

$\binom{4}{1}$: For choosing suit of cards

All orderings $= \binom{4}{1} \dfrac{50!}{(13!)^3 (11!)} + \binom{4}{2} \dfrac{50!}{(13!)^2 (12!)^2}$

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e) All cards are distinct and cards with same suits are together:

Assume cards with same suits as one card so we have 4 cards

Boxes of 13 cards: $\boxed{\heartsuit}\ \boxed{\diamondsuit}\ \boxed{\spadesuit}\ \boxed{\clubsuit} \Rightarrow 4!$ Inside each box: $13!$

f) Aces can not be adjacent:

As we saw in class the best way is ordering all other ranks and use them as wall

All orderings of cards with other ranks:

$$(52-4)! = 48!$$

Now we have 48 cards as wall. we show them by $\underline{\underline{W}}$ and because we have counted orderings of them now we can suppose that those are same

. W W . . . . W

we have 48+1 spots for aces. should not be adjacent so: $\binom{49}{4}$

Putting Aces in Spots

In previous step we ~~supposed~~ find ways for choosing spots for Aces now we have 4 boxes for Aces ⊔ ⊔ ⊔ ⊔ and we can put Aces in 4! different orderings inside these boxes.

All possible orderings: $\underbrace{48!}_{\text{Other ranks orderings}} \times \underbrace{\binom{49}{4}}_{\text{Choosing Spot for Aces}} \times \underbrace{4!}_{\text{Aces ordering}}$

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g) All Spades should be ~~ordered~~ Sorted by rank (Descent or ascent)

Like the question we discussed in class we have restriction.

$$\boxed{\text{① \# of orderings of restricted items}} \times \boxed{\text{② \# of arrangements restricted \& non-restricted items}} \times \boxed{\text{③ \# of orderings of non-restricted items}}$$

① = 2 (Two possible sorting A K Q .. 2 or 2 .. K A)

I ② # of R items: 13 # of NR items: 39

Type I (R) Type II (NR)

13 items 39 items

# of arrangements: $\frac{52!}{(13!)(39!)} = \binom{52}{39}$

③ = 39!

Result = $2 \times \frac{52!}{(13!)(39!)} \times 39!$

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h) First Ace before first King and first 2 before first 3:

$\underbrace{\text{First Ace before first King}}_{R_1}$ and $\underbrace{\text{first 2 before first 3}}_{R_2}$

① Only consider objects affected by R_1: 4 Aces & 4 Kings

we keep one Ace (e.g. spades) and order other cards: $(\underbrace{3}_{\text{Aces}}+\underbrace{4}_{\text{Kings}})!$ ← Because all cards are distinct

but we can keep an Ace by $\underbrace{\binom{4}{1}}_{\text{which Ace we keep}}$ ways

So: $\binom{4}{1} \times (7)!$

② Now consider all items affected by R_1 same and also items affected by R_2

of R_1: 8

" " R_2: 8

$\frac{16!}{8!\,8!} = \binom{16}{8}$

③ Orderings of items with type R_2 $\binom{4}{1}(3+4)!$

④ Items with type R_1 & R_2 (All Aces, kings, 2's, and 3's) → Type R = 16

Other cards: Type NR = 36

# of orderings = $\frac{52!}{36!\,16!}$

⑤ Orderings of non-restricted items: 36!

Result: $\binom{4}{1} 7! \binom{16}{8} \binom{4}{1} 7! \frac{52!}{36!\,16!} 36!$

$\frac{52!}{36!\,16!} = \binom{52}{36}$

I) clubs are sorted by rank and spades $\underbrace{}_{R_1}$ $\underbrace{}_{R_2}$

# of orderings of R_1 × # of arrangements R_1 & R_2 items × # of orderings of R_2 × # of arrangements (R_1+R_2): R_3 & NR × ordering NR

$$\boxed{2 \times \frac{26!}{13!\,13!} = \binom{26}{13} \times 2 \times \frac{52!}{26!\,26!} = \binom{52}{26} \times 26!}$$

J) Only ranks are distinguishable and all Kings and Aces should not be adjacent.

(AK, AA, KK, KA) → not acceptable

Making walls with cards except Aces & kings: (44 cards, 11 Types) $\frac{44!}{(4!)^{11}}$

44 walls ⇒ 45 spots ⇒ Choosing boxes for Aces and kings: $\binom{45}{8}$

Putting Aces & kings inside boxes: (8 cards, 2 Types) $\frac{8!}{(4!)^2}$ → Since only ranks are distinguishable

Result: $\frac{44!}{(4!)^{11}} \binom{45}{8} \frac{8!}{(4!)^2}$

K) Aces should not be adjacent also kings. (All cards are distinct)

First suppose we have not kings in cards so we try to order cards in ways in which Aces are not adjacent: 48 cards (No kings), we put aces aside and make walls with others

$$\boxed{\underbrace{44!}_{\text{\# of possible walls}} \times \binom{\overbrace{44+1}^{\text{spots}}}{\underbrace{4}_{\text{\# of Aces}}} \times \underbrace{4!}_{\text{ordering Aces}}}$$ ⇒ First restriction

Now we have 48 cards (we have counted different orderings, So we assume that those are same)

48 cards are walls for kings $\Rightarrow$ 48+1 spots for 4 kings $\Rightarrow \binom{49}{4}$

# of orderings for kings $= 4!$

Result $= \boxed{44!\binom{45}{4}4!\binom{49}{4}4!}$